Divesting ownership in a rival ☆

Duarte Brito [a,e], Luís Cabral [b,c,*], Helder Vasconcelos [c,d,f]

[a] Faculdade de Ciências e Tecnologia, Universidade Nova de Lisboa, Portugal
[b] New York University, United States
[c] CEPR, United Kingdom
[d] Faculdade de Economia, Universidade do Porto, Portugal
[e] CEFAGE-FCT/UNL
[f] Center for Economics and Finance, Universidade do Porto, Portugal

ARTICLE INFO

ABSTRACT

We examine the consumer welfare effect of a firm's partial ownership of a competitor and compare the implications of alternative forms of divestiture. We identify conditions under which turning voting shares into non-voting shares is preferable to selling the shares to the firm's current shareholders (an option frequently chosen). We also show that selling the voting shares to a large independent shareholder is preferable to selling them to small shareholders. We provide additional theoretical results and apply them to the divestiture of Portugal Telecom's holdings in PTM.

© 2014 Elsevier B.V.

1. Introduction

In 2006, British Sky Broadcasting Group (BSkyB), a UK pay-TV broadcaster, announced the acquisition of 17.9% of ITV, a UK free-to-air TV broadcaster. The UK Competition Commission concluded that such acquisition would lessen competition considerably, and ordered BSkyB to reduce its shareholding to below 7.5%. In a related example, until November 2007 Portugal Telecom (PT) held a 58% share of PT Multimedia (PTM), a combination of voting stock and non-voting stock. The two firms operated in several markets as the two main "competitors" (sometimes the sole competitors). Responding to government pressure that PT divests its share in PT Multimedia, PT's share in PT Multimedia was distributed to PT's shareholders.

These are just two of the many examples where a firm owns a share in a competitor. This situation raises a series of competition policy questions, including: (a) To what extent does partial ownership lessen competition and decrease consumer surplus? (b) What difference does it make whether the partial ownership consists of voting shares, as opposed to preferred (non-voting) stock? (c) If a divestiture of control rights is to take place, what is the best way to implement it: to sell the shares to a large shareholder, to sell the shares to small shareholders, to distribute the shares among the shareholders of the parent company in proportion to their holdings, or to turn the voting stock into preferred stock?

In this paper, we attempt to address question (c), and in the process shed some light on (a) and (b). We propose a basic framework whereby each shareholder cares for his financial interest, whereas each firm maximizes the combined interests of its controlling shareholders.

As a preliminary result, we establish a relation between consumer surplus under a price setting duopoly and the weights that each firm gives to its competitor's profits. We then apply this general result to examine the impact of alternative forms of divestiture. First, we show that turning a partial ownership from voting stock to preferred stock increases consumer welfare. In other words, while a financial interest in a competitor may lessen competition, a controlling share is even worse.

Next, we compare the relative merits — in terms of consumer surplus — of alternative divestiture options. In various recent cases,

divestiture has been implemented by the so-called "proportional" method, whereby firm A's controlling shares in firm B are transferred to the shareholders of firm A in proportion to shareholdings in firm A. We identify conditions under which this option performs worse — in terms of consumer surplus — than turning voting stock into preferred stock, which in turn performs worse than full divestiture (that is, selling the shares to a third party).

Regarding the option of full divestiture, we show that a sale to a large independent shareholder fares better than a sale to many small shareholders. Intuitively, a sale to a large shareholder increases the weight given to independent shareholders in the target firm; and this has the beneficial "countervailing" effect of increasing the weight given by the target firm to its own profit.

While these are our main results, we also provide additional sets of necessary and sufficient conditions to rank various divestiture options. Moreover, while our results are couched in terms of divestiture of partial competitor ownership, they also apply (with the appropriate sign change) to an increase in partial ownership.

1.1. Related literature

A number of authors have considered the impact of partial competitor ownership on the nature of oligopoly competition. In one of the earliest contributions, Reynolds and Snapp (1986) show that market output is lower when there is partial ownership. Bresnahan and Salop (1986) build on Reynolds and Snapp (1986) by introducing the distinction between financial interest and control. They consider a joint venture between two competitors and show that an independent joint venture is more competitive than any form of silent financial interest, which in turn is more competitive than limited joint control or full ownership or control by one parent.[1]

Flath (1992) contributes to this literature by considering both direct (as in the above papers) and indirect financial shareholding. Firm A indirectly holds shares in firm C if it holds shares in firm B and, in turn, firm B holds shares in firm C. The anticompetitive effects are greater in this case than when only direct holdings are considered.[2]

In a recent contribution, Karle et al. (2011) consider a private investor who initially owns a controlling stake in one of two competing firms and may acquire a (controlling or non-controlling) stake in the competitor, either directly (by making use of own funds) or indirectly (by inducing the controlled firm to do so). While there is some overlap with our analysis, their framework cannot be used to address the question we are interested in this paper, namely comparing various forms of divestiture.[3]

Although most of the literature focuses on unilateral effects of partial ownership, Gilo et al. (2006) look at the possibility of coordinated effects. Specifically, they analyze whether passive financial investments in rivals facilitate or hinder tacit collusion. Despite the fact that larger crossholdings may limit the punishment after deviation from a collusive arrangement (Malueg, 1992), Gilo et al. (2006) establish that an increase in financial ownership by a rival firm never hampers collusion.

The paper that is closest to ours is O'Brien and Salop (2000). They study the case when there is partial ownership which may or may not correspond to control. They evaluate the impact of such cross shareholdings by computing each firm's price pressure index (PPI): an increase in firm i's PPI corresponds to an upward shift in its first-order condition; given constant rival prices, this leads to a higher price by firm i. Based on this methodology, they find the surprising result that obtaining control of a rival firm through partial ownership may be worse, in terms of welfare, than a complete merger between the two competitors.[4] Some of our results are consistent with those of O'Brien and Salop (2000). However, our framework allows us to consider additional ownership comparative statics they did not consider.

O'Brien and Salop focus on partial acquisitions that lead to various scenarios. However, the relationship between financial interest and control is not modeled. By distinguishing between voting stock and preferred stock, our approach addresses this issue and derives a series of policy relevant results. Moreover, unlike O'Brien and Salop we allow explicitly for the distinction between individuals as owners and firms as owners, raising the issue of direct and indirect control or financial interest. As our empirical application shows, this distinction is of practical interest.

1.2. Road map

The remainder of the paper is structured as follows. In Section 2 we present our formal framework. Section 3 includes some preliminary results (lemmas) which we then use in Section 4, where we present our main results. An extension to our basic framework, considering the case of common shareholders, is included in Section 5. In Section 6, we apply our analysis to the case of Portugal Telecom's (PT) divestiture of its share in PT Multimedia (PTM). Section 7 concludes the paper.

2. Formal approach

Consider an industry with two firms (A and B) and N relevant shareholders.[5] We explicitly consider the distinction between voting stock (i.e., shares with control rights) and preferred (non-voting) stock. Firm i's total stock ($i = A,B$) is composed of a percentage V_i of voting stock and a percentage $1 - V_i$ of preferred stock. Shareholder n holds a share v_{in} of voting stock in firm i and a share s_{in} of preferred stock in firm i. Hence, shareholder n holds a percentage $t_{in} \equiv v_{in}V_i + s_{in}(1 - V_i)$ of firm i's total stock.

Each firm's profit is distributed among shareholders proportionally to their total stock, regardless of whether it be voting stock or preferred stock. Hence, shareholder n receives a profit stream corresponding to a percentage t_{in} of firm i's aggregate profit, Π_A. It follows that shareholder n's payoff is given by $t_{in} \Pi_i + t_{jn} \Pi_j$.

In addition to individual shareholders, we also consider the possibility that firm A owns a share t_{B0} in firm B, which includes a share $v_{B0} > 0$ of voting stock.[6] It follows that, if π_i is firm i's operating profit ($i = A,B$), then firm A's aggregate profit (including cross-holdings) is given by $\Pi_A = \pi_A + t_{B0}\Pi_B$, whereas for firm B we have simply $\Pi_B = \pi_B$.

We follow O'Brien and Salop (2000) in assuming that each firm's objective function is a weighted sum of shareholders' payoffs. Additionally, we assume that the weight given by firm i to shareholder n's payoff, w_{in}, is a function of shareholder n's voting stock. In particular, let $w_{in} = f(v_{in})/\sum_{n=0}^{N} f(v_{in})$, where: (i) $f(0) = 0$; (ii) $\partial w_{in}/\partial v_{in} > 0$; (iii) $f(v_{in})/f(v_{in'}) = f(\theta v_{in})/f(\theta v_{in'})$ for all $\theta \neq 0$ and (iv) $f(\theta v_{in}) + f((1 - \theta)v_{in}) \leq f(v_{in})$ for all $\theta \in [0,1]$. We thus assume that a firm gives no weight to a

[1] Reitman (1994) considers the same ownership structure as Reynolds and Snapp (1986) in a conjectural variation model to discuss the incentives firms may have in participating in partial ownership arrangements. See also Alley (1997) for an application of a conjectural variation model with partial ownership arrangements and trade to the automobile industry.

[2] Dietzenbacher et al. (2000) extend these results to more than three firms and to Bertrand competition. They also provide an empirical application to the Dutch financial market. In related recent research, Micola and Bunn (2008) conducted a series of simulations to analyze the effects of crossholdings on the outcome of sealed bid-offer auctions with capacity constraints.

[3] Moreover, Karle et al. (2011) only consider two possible extreme cases regarding initial ownership structures in the target firm: one block holder or many small shareholders. In addition, all private investors are assumed not to have initially positions in more than one firm. Our present paper proposes a more general framework in both respects, which is important in terms of empirical application.

[4] In a recent contribution Foros et al. (2010) consider the case when the partial ownership of one firm in the other is endogenously determined.

[5] We allow for two types of shareholders, relevant shareholders and infinitesimal shareholders. Only the former are able to influence the firms' managers.

[6] Strictly speaking, firm B has, at most, $N + 1$ shareholders, if we include the competing firm as a shareholder.

particular shareholder who does not own voting stock, that the weight given to a particular shareholder is increasing in his percentage of voting stock, that the ratio of weights corresponding to any two shareholders, n and n', does not change if the percentage of voting stock owned by each of them changes in the same proportion and that if a given shareholder's stock is split between two shareholders, the weight given by firm i to these two shareholders is lower than the original weight.

Notice that in our setting, even if shareholder n's percentage of voting stock is kept constant, if for some reason there is a change in the pattern of ownership of the remaining shareholders, this may affect the weight attributed by the firm to shareholder n's payoff. Moreover, even if one shareholder has the majority of the votes we do not consider that the firm maximizes his payoff. As highlighted by O'Brien and Salop (2000) "the control exerted by shareholders with the majority of voting power is not absolute. If it were, the minority (or non-voting) shares would have little value…" (p. 571).

Formally, firm i maximizes ω_i ($i = A,B$), which are given as follows:

$$\omega_A = \sum_{n=1}^N w_{An}(t_{An}\ \Pi_A + t_{Bn}\ \Pi_B)$$
$$\omega_B = \sum_{n=1}^N w_{Bn}(t_{An}\ \Pi_A + t_{Bn}\ \Pi_B) + w_{B0}\ \omega_A. \tag{1}$$

Solving for ω_A and ω_B and expressing aggregate profits in terms of operational profits, we obtain:

$$\omega_A = \sum_{n=1}^N w_{An}(t_{An}\ \pi_A + (t_{Bn} + t_{An}t_{B0})\pi_B)$$
$$\omega_B = \sum_{n=1}^N (w_{Bn} + w_{B0}w_{An})(t_{An}\ \pi_A + (t_{Bn} + t_{An}t_{B0})\pi_B). \tag{2}$$

Thus, an alternative way to think about firm i's maximization problem is to consider that, ultimately, there are N shareholders with control rights (through w_{in}) and financial interests (through t_{in}) in firms A and B, both direct and indirect interests. For instance, the direct financial interests of shareholder n in firm B correspond to t_{Bn}, whereas his indirect financial interests are given by $t_{An}t_{B0}$. A similar reasoning can be applied to the control weights w_{in}.

Many of our results below are better expressed in terms of share correlation/concentration indexes. Specifically, we define

$$K_{ij} \equiv \sum_{n=1}^N (w_{in} + w_{i0}w_{hn})t_{jn} \tag{3}$$

with $h,i,j = A,B$ and $i \neq h$. In words, K_{ij} measures the correlation between direct plus indirect control in firm i and financial interest in firm j, as well as the concentration of such control/interest. In the limit, $K_{ij} = 0$ either because firm i's control is separated from firm j's ownership or because there are many infinitesimal shareholders. At the opposite extreme, $K_{ii} = 1$ if there is a single shareholder who controls the firm and is entitled to all its profit.

With this notation, both firms' objective functions can be written as

$$\omega_A = K_{AA}\pi_A + (K_{AB} + K_{AA}t_{B0})\pi_B$$
$$\omega_B = K_{BA}\pi_A + (K_{BB} + K_{BA}t_{B0})\pi_B. \tag{4}$$

Without loss of generality and assuming $K_{ii} > 0$, one can re-write the firms' objective functions as

$$\omega_A = \pi_A + \gamma_A\pi_B$$
$$\omega_B = \pi_B + \gamma_B\pi_A \tag{5}$$

where

$$\gamma_A = t_{B0} + \frac{K_{AB}}{K_{AA}}$$
$$\gamma_B = \frac{K_{BA}}{K_{BB} + K_{BA}t_{B0}} \tag{6}$$

correspond to the weights each firm attributes to its competitor's operating profit.

3. Preliminary results

We are interested in understanding how changes in ownership affect consumer welfare. For this purpose, we assume that firms compete in prices and that prices are strategic complements: $\partial^2\pi_i/\partial p_i\partial p_j > 0$. Moreover, we assume that an increase in firm i's price increases firm j's operating profit π_j and reduces consumer surplus. The following result then relates changes in γ_i to changes in consumer surplus.

Lemma 1. *Consumer surplus is decreasing both in γ_A and γ_B.* ∎

Although we focus our attention on price competition, the negative relation between γ_i and consumer surplus extends to other modes of market competition (for example, Cournot competition with linear demands). The idea is fairly intuitive: to the extent that firm i takes firm j's profit into consideration when maximizing profits, we are closer to the perfect collusion extreme, which implies higher prices.

Although our focus is on changes in firm A's holdings of firm B, we first consider a series of comparative statics exercises with respect to infinitesimal changes in some key parameters.

Lemma 2. *γ_B is increasing in K_{BA}, non-increasing in K_{BB} and decreasing in t_{B0}. Moreover, γ_A is increasing in t_{B0} and K_{AB}; and non-increasing in K_{AA}.* ∎

The weight placed by firm i on its rival's profit decreases with K_{ii}. Whenever control (measured by direct and indirect control weights) and financial interests become more aligned in firm i, shareholders will use their control to make firm i maximize own profit and thus place a lower weight on the rival's profit.[7] For a similar reason, the weight placed by firm i on its rival's profit increases with K_{ij}. If shareholders with (direct or indirect) control of firm i have more financial interests on firm j, then firm i will place a larger weight on its competitor.

As for firm A's financial interests in firm B, t_{B0}, it affects both firms in different ways. An increase in t_{B0} makes firm A care more about firm B's profits due to its larger financial interest in firm B. How about firm B? To the extent that $K_{BA} > 0$, i.e. that there is some correlation between control in firm B and financial interest in firm A, a larger t_{B0} makes firm B place a higher weight on own profit and consequently a lower profit on rival's profit. Everything else constant, the more firm A cares about firm B (as measured by t_{B0}), the more firm A will use its control over firm B to maximize firm B's profits, which in turn leads to a lower γ_B.

We now discuss in more detail the factors that drive K_{ij}. For that purpose, we define

$$k_{ij} \equiv \sum_{n=1}^N w_{in}t_{jn} \tag{7}$$

$i = A, B$, so that we can write

$$K_{ij} \equiv \sum_{n=1}^N (w_{in} + w_{hn}w_{i0})t_{jn} = k_{ij} + w_{i0}k_{hj}$$

---

[7] It should be noted that $\partial\gamma_i/\partial K_{ii}$ is negative when $K_{ij} > 0$. Otherwise it is zero. Recall that when K_{ij} is zero, shareholders with control of firm i have no participation in firm j's profit. As long as $K_{BB} > 0$, firm B will maximize own profit and as long as $K_{AA} > 0$, firm A will maximize its aggregate profit, which includes a percentage t_{B0} of firm B's operating profit. Increases in K_{AA} or K_{BB} do not change this behavior.

with $h,i,j = A,B$ and $i \neq h$. Again, k_{ij} measures the correlation between control in firm i and financial interest in firm j, as well as the concentration of such control/interest. But in this case, it refers exclusively to direct control.[8]

With respect to firm A, there is only direct control, and hence, $K_{Aj} = k_{Aj}$. There exist, however, two types of control regarding firm B: direct control and indirect control (which is exerted via firm A). Hence, K_{Bj} is composed of two elements. The first one measures the extent to which direct control in firm B is correlated with financial interest in firm j, k_{Bj}. The second one measures the extent to which control in firm B exerted via control in firm A is correlated with financial interest in firm j, $w_{B0}k_{Aj}$. In what follows, we will assume that direct control in a given firm is more correlated with (direct) financial interest in the same firm than with financial interest in the rival firm, which implies that $k_{AA}k_{BB} - k_{AB}k_{BA} > 0$. Under this assumption, lemma 2 can then be rewritten as:

Lemma 2′. γ_B is increasing in k_{AA}, k_{BA} and v_{B0}; and decreasing in k_{BB}, k_{AB} and t_{B0}. Moreover, γ_A is increasing in t_{B0} and k_{AB}; and non-increasing in k_{AA}.■

As far as γ_A is concerned, since $K_{AA} \equiv k_{AA}$ and $K_{AB} \equiv k_{AB}$ (see footnote 8), the results in Lemma 2′ coincide with those in Lemma 2. With regard to the part that refers to γ_B, however, this formulation allows us to highlight further effects, which we discuss in turn.

First, γ_B is increasing in k_{AA}. Why would firm B care about firm A's profits? For two reasons: (*i*) because some shareholders with control in firm B's may have a financial interest in firm A (i.e. because there exist common shareholders); and/or (*ii*) because the shareholders with control and financial interest in firm A also have some control in firm B (through w_{B0}). The extent to which the shareholders who control firm A also have a financial interest in firm A is measured by k_{AA}, thus γ_B is increasing in k_{AA}.

Second, γ_B is decreasing in k_{BB}. Recall that the indexes k_{ij} measure both correlation and concentration. In this case, it is best to think of k_{BB} as a *concentration* index. In the limit when k_{BB} is small, all firm B shareholders (except firm A) are of infinitesimal size. This implies that firm A's share in firm B, even if less than 1, effectively gives firm A complete control over firm B. To the extent that k_{AA} is greater than zero, so that firm A's voting shareholders also have a financial interest in firm A, this implies that the controlling share v_{B0} will induce firm B to care a lot for firm A's profits. For this reason, an increase in k_{BB} — a greater concentration of control by firm B's "independent" shareholders — "counterweights" the effect of v_{B0}. We will return to this "countervailing" effect later.

Third, γ_B is increasing in v_{B0} (and in w_{B0}) if $k_{AA}k_{BB} - k_{AB}k_{BA} > 0$. Greater control of firm B by firm A's shareholders leads firm B to place greater weight on firm A's profits. There are two opposing effects in place: (*i*) A larger w_{B0} increases the correlation between control in firm B and financial interest in firm A, K_{BA}, via indirect control; and (*ii*) a larger w_{B0} increases the correlation between control in firm B and financial interest in firm B, K_{BB}, if k_{AB} is positive. The channel through which the first effect takes place is similar to that discussed above regarding the partial derivative $\partial\gamma_B/\partial k_{AA}$. To the extent that $k_{AA} > 0$, the voting shareholders who have partial indirect control over firm B also have a financial interest in firm A. Such control is used to influence firm B to give greater weight to firm A's profits. And the greater v_{B0} is, everything else constant, the greater the control of firm B by firm A, and the greater this effect is. The second effect results from the possible existence of common shareholders who control firm A and have a financial interest in firm B. To the extent that $k_{AB} > 0$, some shareholders are interested in making firm B place a larger weight on own profit and an increase in w_{B0} makes their indirect control of firm B stronger.

Notice the contrast between two of the derivatives: γ_B is increasing in v_{B0} but decreasing in t_{B0}. This shows that the nature of firm A's holdings in firm B matters a great deal: control and financial interest are related but different forms of shareholding. In fact, for a given level of financial interest, greater control leads firm B to place greater weight on firm A's profits; but for a given level of control, greater financial interest leads firm B to place a lower weight on firm A's profits. In the next section we will discuss the importance of this distinction when comparing various forms of divestiture.[9]

Discussing the effects of partial ownership, O'Brien and Salop (2000) also allude to the important distinction between active and passive ownership:

> In analyzing the competitive effects of partial ownership, it is necessary to distinguish between two aspects of partial ownership, financial interest and corporate control … These two factors have separate and distinct impacts on the competitive incentives of the acquired and acquiring firm. Financial interest affects the incentives of the acquiring firm, while corporate control affects the incentives of the acquired firm. (p. 568)

While we agree with this characterization, we also think that it is incomplete. We agree that a higher t_{B0} leads to a higher γ_A ("financial interest affects the incentives of the acquiring firm") and that a higher v_{B0} leads to a higher γ_B ("corporate control affects the incentives of the acquired firm"). But to this we add that, to the extent that firm A has some control over firm B, financial interest also affects the incentives of the acquired firm (a higher t_{B0} leads to a lower γ_B, as stated in Lemma 2).

In addition to the derivatives presented above, it is also relevant to discuss how γ_A and γ_B are affected by k_{AB} and k_{BA}. With respect to k_{AB}, a higher correlation between (direct) control in firm A and financial interest in firm B makes firm A place a higher weight on firm B's profit and, to the extent that the shareholders who control firm A have an influence via w_{B0} on firm B's decisions, will make firm B place a higher weight on itself, i.e. a lower weight on firm A's profit. As for k_{BA} we have $\partial\gamma_A/\partial k_{BA} = 0$ and $\partial\gamma_B/\partial k_{BA} > 0$. The reasons are similar, with the exception that firm B has no voting shares in firm A and, hence, shareholders who control firm B have no way to instruct firm A to place a higher weight on its own profit. As a result of the above, the existence of common shareholders leads to a higher γ_A, whereas the effect on γ_B is uncertain. If $k_{BA}k_{BB} - k_{AB}k_{AA}w_{B0}^2 > 0$, the existence of common shareholders also leads to a higher γ_B.

Since in the presence of common shareholders, changes in the ownership and control structure give rise to several effects going in opposite directions, it is difficult to establish general results. In the following section, we present our main results for the case in which there are no common shareholders before the divestiture. Section 5 then discusses the implications on the main results of the existence of common shareholders prior to the divestiture.

4. Main results: no common shareholders

Although our general notation allows for shareholders with holdings in both firms, in this section we will assume that this is not the case except for the share firm A holds in firm B. Taking this into account, we have $k_{AB} = k_{BA} = 0$ and we may rewrite Eq. (6) as follows:

$$\gamma_A = t_{B0}$$
$$\gamma_B = \frac{w_{B0}k_{AA}}{k_{BB} + w_{B0}t_{B0}k_{AA}} . \tag{8}$$

---

[8] Given that we assume that only firm A has some financial interest and control in firm B, we have $w_{A0} = 0$ and the expressions simplify to $K_{AA} \equiv k_{AA} > 0$, $K_{AB} \equiv k_{AB} \geq 0$, $K_{BB} \equiv k_{BB} + w_{B0}k_{AB} > 0$ and $K_{BA} \equiv k_{BA} + w_{B0}k_{AA} > 0$.

[9] As we will see, the two forms of shareholding are not independent: an increase in voting shares also leads to an increase in financial interest, except in the limit case when $V_i = 0$, so that voting shares correspond to no financial interest.

We now turn to the main results in the paper, where we consider the implications for consumer surplus of various alternative forms of divestiture of firm A's control holding in firm B. Specifically, using Lemmas 1, 2 and other results, we now characterize the effect and compare the relative merits in terms of consumer surplus of the following alternative options:

1. The shares are distributed among the shareholders of the parent company in proportion to their holdings.
2. The shares are turned into preferred stock.
3. The shares are sold to infinitesimally small shareholders.
4. The shares are sold to a new large shareholder.

Note that there is an important distinction between Options 1 and 2, on the one hand, and Options 3 and 4 on the other hand: the latter two options require a sale of shares, whereas the first two do not. In other words, one would expect the transactions cost of Options 3 and 4 to be greater than that of Options 1 and 2. We will return to this issue when discussing the relative merits of each option.

Option 1 assumes that all shares (irrespective of being voting or preferred stock) that firm A owns in firm B are distributed to firm A's shareholders in proportion to their total stock in the parent company. If there is only one type of shares, then there is only one type of proportional spin-off. If, however, there are various types of shares, then our assumption is not innocuous, as it is likely that different rules will lead to different outcomes. For example, each share might be distributed in proportion to the shares of the same type held in the parent company.[10]

Option 2 directly addresses the issue of control. If the main concern is that firm A controls firm B, then the simplest way of addressing the issue is to remove such control by turning its voting shares into non-voting shares.

Option 3 is a natural benchmark. Just as the most competitive market structure corresponds to atom-sized firms, one may conjecture that atom-sized shareholders best correspond to the idea of a competitive structure. However, as we will later see, such shareholder structure does not necessarily maximize consumer surplus.

Finally, Option 4 is similar to Option 3 with the difference that the sale is made to one large shareholder.

We first consider the case of a proportional spin-off procedure (Option 1). In some sense, a proportional divestiture substitutes direct firm control for shareholder control. To the extent that these alternative forms of control are substitutes, a tantalizing possibility is that a proportional divestiture is neutral from the point of view of effective control, and therefore from the point of view of consumer welfare. However, redistributing all of firm A's shares in firm B to existing firm A's shareholders in proportion to their shareholdings in firm A, t_{An}, is, in general, not neutral: the proportional spin-off may increase or decrease consumer welfare depending on the particular patterns of ownership structure. This is for two reasons. First, since the indirect voting shares are converted into direct shares in proportion to t_{An} and not in proportion to v_{An}, the relative voting importance of each shareholder changes, thereby affecting the shareholder's weight even if the latter is linear in voting stock. Second, even if $v_{An} = t_{An}$, as the weight assigned to each shareholder is not necessarily linear in the percentage of voting stock she holds, the importance of each shareholder may change: a given reduction in the indirect percentage of votes is not necessarily compensated

by an equal increase in the direct percentage of votes held, as far as control weights are concerned.

This type of divestiture has several effects, which we discuss in turn.

First, it creates common shareholders. If $k_{AA} > 0$, some shareholders with control over firm A will, after the divestiture, have a financial interest in firm B (i.e. k_{AB} becomes positive). This will lead firm A to attribute a higher weight to firm B's profit (i.e. γ_A increases). On the other hand, to the extent that, after the divestiture, firm A still retains some control over firm B, these common shareholders will use their control over firm A to make firm B more aggressive, thereby leading to a decrease in γ_B. Additionally, the shareholders who gain some direct control over firm B also have a financial interest in firm A (i.e. k_{BA} becomes positive). Hence, these shareholders will make firm B behave less aggressively (i.e. a larger γ_B).

Second, the proportional divestiture leads to a reduction in both v_{B0} and t_{B0}, where the corresponding induced effects on γ_A and γ_B were already discussed in the previous section (see Lemma 2 and Lemma 2′).

Finally, the proportional divestiture also has an impact on k_{BB}. By redistributing shares that were previously concentrated in firm A to its own shareholders, this type of divestiture leads to an increase in the correlation between ownership and control in firm B, which has the positive effect of making this firm place a lower weight on firm A's profit (i.e. γ_B is reduced).

Before presenting our next result, which establishes a necessary and sufficient condition for the proportional divestiture to be good for consumers, some additional notation must be introduced. Define

$$S_{ij}^o \equiv \sum_{n=1}^{N} \Delta w_{in}^o \, t_{jn}$$

where Δw_{in}^o denotes the variation in w_{in} due to the implementation of Option $o = 1, \ldots, 4$ divestiture. In words, S_{ij}^o refers to the correlation between variations in control weights in firm i (resulting from divestiture o) and original ownership shares in firm j. One particular case is worth explaining. We have $S_{ij}^o = 0$ when each shareholder who is given a different weight in firm i's objective function did not own directly any stock in firm j before the divestiture (and hence had no direct claims over firm j's profit). A large S_{ij}^o means that shareholders who gained more control over firm i are those who have, at the outset, a larger financial interest in firm j.

Proposition 1. *Any complete or partial proportional divestiture (Option 1) increases consumer welfare if and only if*

$$k_{BB}\left(S_{BA}^1 + k_{AA}\Delta w_{B0}^1\right) - S_{BB}^1 w_{B0} k_{AA} < 0.$$

∎

The first term in the condition for consumer surplus to increase can be written as $k_{BB} \sum_{n=1}^{N} (\Delta w_{Bn}^1 + w_{An}\Delta w_{B0}^1)t_{An}$. If this term is negative, Option 1 increases consumer welfare. This term is negative if the largest shareholders in terms of financial interest in firm A are the ones whose weights in firm B's objective function (both direct and indirect weights) decrease with the divestiture. In fact, shareholders with $\Delta w_{Bn}^1 + w_{An}\Delta w_{B0}^1 < 0$ will see their overall weight on firm B's objective function decrease, because the increase in direct weight, Δw_{Bn}^1, does not compensate for the decrease in their indirect influence in firm B, $w_{An}\Delta w_{B0}^1$. If these shareholders, the ones whose relative voting power in firm B decreased with the divestiture, also have a high financial interest in firm A, a high t_{An}, they will now be less able to use their (voting) influence in firm B so as to instruct it to give more weight to firm A, thereby reducing competition and hurting consumer surplus. In case, however, this term is non-negative, it is still possible that Option 1 increases consumer welfare, provided that S_{BB}^1 is sufficiently large. Notice that the original firm B shareholders will see their degree of control increase as the voting stock of firm B owned by firm A becomes diluted after the divestiture, $S_{BB}^1 > 0$.

---

[10] There are a number of examples consistent with our particular assumption regarding proportional spin-offs: (i) Brookfield Asset Management Inc.'s 2008 spin-off of Brookfield Infrastructure Partners L.P. (where each holder of Brookfield Class A and Class B shares received one share for each 25 Brookfield shares held); (ii) Bedminster National Corp.'s 2007 spin-off of its two subsidiaries, Bedminster Capital and Bedminster Financial (where holders of common stock of Bedminster National received one share of the Class A and Class B common stock of each subsidiary for every share of Bedminster National Class A and Class B common stock held); and (iii) NACCO's proposed spin-off of Hamilton Beach (where for each share of NACCO Class A or Class B common stock held, NACCO distributed one half of one share of Hamilton Beach Class A common stock and one half of one share of Hamilton Beach Class B common stock).

In order to illustrate in a simpler way the main mechanisms behind this result, consider the special case wherein the $f(\cdot)$ function is linear. More specifically, assume now that $f(v_{Bn}) = v_{Bn}$ and $\sum_{n=0}^{N} v_{Bn} = \sum_{n=1}^{N} t_{An} = 1$. When this is the case, the condition in Proposition 1 boils down to

$$\sum_{n=1}^{N} (t_{An} - v_{An}) t_{An} < 0.$$

In this case, the condition for consumer welfare to increase will be verified if the largest shareholders in terms of financial interest are also the ones who have a lower percentage of total stock when compared to their percentage of voting stock. Shareholders with $t_{An} > v_{An}$ will be attributed a higher direct percentage of the divested voting shares, $t_{An} v_{B0}$, than their ex-ante indirect voting influence in firm B, $v_{An} v_{B0}$. Now, if these shareholders, the ones whose relative voting power in firm B increases with the divestiture, also have a high financial interest in firm A, a high t_{An}, they will use their increased (voting) influence to instruct firm B to be less aggressive.

Consider the following illustrative example: two shareholders, 1 and 2, are each entitled to 40% and 60% of firm A's profit, respectively, but one of them (say, shareholder 1) owns all voting stock. Further, firm A owns 20% of firm B's voting stock. Before the divestiture, shareholder 1 has an indirect control over firm B that corresponds to 20% (i.e. firm B will give a weight of 20% to shareholder 1's payoff) and the remaining shareholder has no control over firm B, i.e. is given no weight by firm B. After the proportional divestiture, 12% of firm A's voting stock in firm B will be attributed to shareholder 2 and 8% to shareholder 1. In this case, $\sum_{n=1}^{N} (t_{An} - v_{An}) t_{An} = .12 > 0$, meaning that the proportional divestiture will reduce consumer surplus. Shareholder 2 will see his control in firm B increase when compared to shareholder 1. This happens because shareholder 2 will be given a larger percentage of voting stock, 12%, than his original indirect holdings, 0%, while shareholder 1 will receive only 8% and previously held 20%. But, given that shareholder 2 is more interested in firm A's profit than shareholder 1 was (because he has a larger financial interest) the increase in control will make firm B less aggressive.

It is worth remarking at this point that with $f(v_{Bn}) = v_{Bn}$, if each shareholder n owns the same percentage of voting shares and preferred stock in firm A (i.e., $v_{An} = s_{An} = t_{An}$), then a proportional divestiture turns out to be neutral from a consumer welfare point of view.[11] The intuition is straightforward: the proportional divestiture simply converts indirect voting power and financial interest into direct ones. The same is true if there is only one type of stock.

Next, we consider the possibility of turning firm A's voting stock in firm B into preferred stock (Option 2).

Proposition 2. *Any complete or partial divestiture that turns voting stock v_{B0} into preferred stock (Option 2) leads to an increase in consumer surplus.*∎

Turning voting stock into preferred stock does not completely separate the firms: the financial interest is kept at the same level. However, as far as consumer surplus is concerned, it represents a positive move. As shown by Lemma 2′, a decrease in v_{B0} leads to a decrease in γ_B; and this in turn leads to an increase in consumer surplus, as shown in Lemma 1. Moreover, a decrease in v_{B0} leads to an increase in k_{BB}. This happens because the independent voting shares v_{Bn} now have relatively greater value. And, by Lemma 2′, an increase in k_{BB} leads to a decrease in γ_B.

Intuitively, the switch from voting stock to preferred stock has two effects on control: it weakens firm A's control of firm B (the direct effect) and it also strengthens control by firm B's independent shareholders.

Both of these effects lead firm B to place a lower weight on firm A's profit.

Currently competition policy in the EU and the U.S. seems roughly consistent with Proposition 2's characterization of the effects of partial ownership. In the EU, any partial interest that enables the purchaser to exercise control over the target company is subject to a merger filing. The same is not true, however, if the acquisition does not change the degree of control. The U.S. guidelines also make a distinction between active and passive acquisitions, though they also recognize that even passive acquisitions may present competitive concerns (Hatton and Cardwell, 2000).

The following proposition considers the situation in which there is a divestiture of voting stock v_{B0} that takes place by sale (Options 3, 4).

Proposition 3. *A complete divestiture of voting stock v_{B0} that takes place by sale (Options 3, 4) leads to an increase in consumer surplus.*∎

The intuition behind this result is simple. When a complete divestiture of voting stock takes place by sale, then, after the sale, $v_{B0} = w_{B0} = 0$. From Eq. (8), this leads to firm B attributing *no weight* to firm A's profits, $\gamma_B = 0$. In addition, t_{B0} will decrease by $v_{B0} V_B$, implying that also γ_A (the weight given by firm A to firm B's profits) will decrease with the complete divestiture by sale (Lemma 2).[12] Now, since both γ_A and γ_B decrease with the operation, Lemma 1 implies that consumer surplus will be enhanced after a complete divestiture by sale.[13]

It should be noted, however, that the divestiture of a small fraction of v_{B0} by means of a sale to an infinite number of small shareholders may lead to a decrease in consumer surplus. In order to illustrate this possibility, consider the following simple example in which (i) all shares are voting shares, (ii) there are no common shareholders and (iii) $f(v_{in}) = v_{in}$. Following Singh and Vives (1984) and assuming firm symmetry, let the representative consumer maximize

$$U(q_A, q_B) = a(q_A + q_B) - \frac{1}{2}\left(bq_A^2 + bq_B^2 + 2dq_Aq_B\right) - p_A q_A - p_B q_B$$

where a, b and d are positive parameters, with $b > d$, and q_i denotes firm i's output. This utility function leads to the following linear inverse demand function:

$$q_i = \frac{a(b-d) - bp_i + dp_j}{b^2 - d^2}.$$

Assuming that marginal costs are equal to zero, it is straightforward to obtain normalized equilibrium prices and outputs, which are presented in Appendix B.

Consumer surplus, if divided by a^2/b, can be written as

$$CS^* = (Q_A + Q_B) - \frac{1}{2}\left(Q_A^2 + Q_B^2 + 2xQ_AQ_B\right) - P_AQ_A - P_BQ_B$$

where Q_i and P_i are normalizations of equilibrium quantities and prices and $x = d/b < 1$.

Now, the divestiture of a small fraction of v_{B0} leads to a decrease in consumer surplus if

$$\frac{\partial CS^*}{\partial v_{B0}} = \frac{\partial Q_A}{\partial v_{B0}}(1 - Q_A - xQ_B - P_A)$$
$$+ \frac{\partial Q_B}{\partial v_{B0}}(1 - Q_B - xQ_A - P_B) - \frac{\partial P_A}{\partial v_{B0}}Q_A - \frac{\partial P_B}{\partial v_{B0}}Q_B > 0.$$

---

[11] It is also needed that $\sum_{n=1}^{N} t_{An} = 1$ so that all of firm A's stock in firm B is attributed to relevant shareholders.

[12] Recall that with no common shareholders, firm A cares about firm B's profits to the (precise) extent that it has a financial interest in firm B, $\gamma_A = t_{B0}$, which follows from Eq. (8).

[13] If the sale included all types of stock, and given that we are assuming no commons shareholders, it would separate firms completely: $\gamma_A = \gamma_B = 0$.

The following figure represents in the (v_{B0}, x) space the area for which this derivative is positive, for small arbitrary values of $k \equiv k_{BB}/k_{AA}$. For each of three arbitrary values of k, the previous condition holds below the corresponding curve. Appendix B provides the details.

To understand this effect, note that after a divestiture of firm A's holdings in firm B by means of a sale to an infinite number of small shareholders, the values of k_{AA} and k_{BB} remain constant.[14] From Eq. (8), we conclude that the only effect in γ_A and γ_B is through v_{B0} and t_{B0} — or simply v_{B0}, since $t_{B0} = v_{B0}$ given our assumption that all shares are voting shares.

When v_{B0} (firm A's share of firm B) decreases with the sale, so does the value of γ_A (the weight placed by firm A's management on firm B's profits). This is fairly intuitive and reflects the first equation in Eq. (8): the weight given by firm A to firm B's profits is proportional to firm A's financial interest in firm B. (This is also part of Lemma 2'.)

What is perhaps not as intuitive is that a reduction in v_{B0} leads to an *increase* in the weight given by firm B to firm A's profits. In the limit, suppose that there is an infinite number of infinitesimal shareholders in firm B, except for firm A, which holds a strictly positive share, i.e. that k_{BB} is very small. This means that firm A has effective control of firm B, that is, firm B's objective is effectively firm A's objective. If we now reduce firm A's holding in firm B to a low level, then firm A's financial interest in firm B is also reduced. In the limit as this value becomes close to zero, firm A's interest becomes identified with firm A's profits. Since firm A has effective control over firm B, it follows that firm B's management becomes more concerned with firm A's profits, a change that corresponds to an increase in γ_B. The areas identified in Fig. 1 illustrate that the above increasing effect in γ_B may be sufficiently strong to counteract the "natural" decreasing effect in γ_A. Specifically, the condition that k is sufficiently small implies that firm A's voting share in firm B, v_{B0}, effectively gives firm A control of firm B.

O'Brien and Salop's (2000) comments on the effects of partial ownership echo the possible effects described in this example:

> Partial investments can raise either larger or smaller concerns than complete mergers. This may seem surprising, since partial acquisition would appear to align the parties' interests less in all cases than would a complete merger. The competitive effects of partial ownership depend critically on two separate and distinct elements: financial interest and corporate control. This distinction is absent in merger analysis, which assumes that the acquiring firm (or person) automatically controls the acquired entity after the merger. With partially ownership interests, however, these elements are separable. They can occur in ways that result in greater or lower harm to competition than a complete merger. (p. 562)

In fact, re-interpreting Proposition 3 as a result about acquisitions (that is, changing "signs"), we obtain a situation very much like the one characterized by O'Brien and Salop's (2000). A full acquisition may not be as bad as a partial acquisition to the extent that the full merger does not increase control by much but, by increasing firm A's financial interest in firm B, lowers the relative weight that firm B places on firm A's profits. In other words, as O'Brien and Salop (2000) rightly point out, the effects of partial ownership depend critically on the balance between financial interest and corporate control. Notice moreover that, as suggested by O'Brien and Salop and stressed by Foros et al. (2010), the above line of argument implies that, under some conditions, there may be ownership structures worse than a full merger. In other words, starting from a full merger, the sale of some of firm B's stock owned by firm A may lead to a decrease in consumer surplus. In our framework, this corresponds to starting with the case $v_{B0} = t_{B0} = 1$ and $k_{BB} = 0$ and considering a small reduction in t_{B0}.
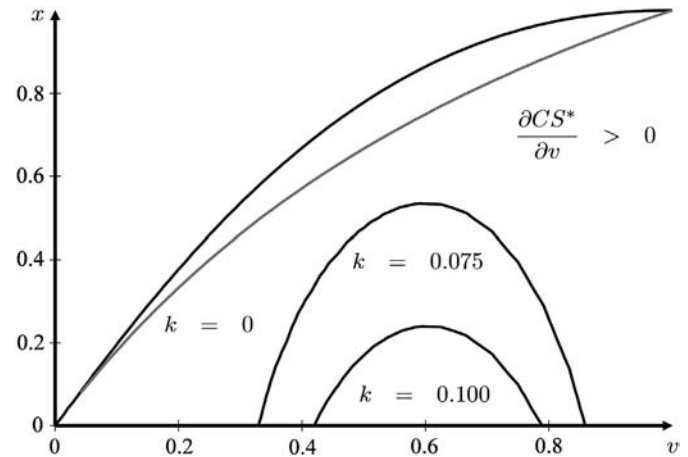

Fig. 1. Consumer surplus effects of the divestiture of a small fraction of v_{B0}.

How can one possibly go from full merger to something that is even worse than full merger? The idea is that when a firm has control over the prices of two substitutes, it has the incentive to increase one of them so as to increase the demand for the other one. In so doing, it will face a trade-off between higher profits in one of the firms and lower profits in the other one. To the extent that the controlling firm cares less for the profit of the target firm — because it does not completely own it — it has incentive to increase prices more than in the case of a complete merger.

Our final results in this section compare the relative merits, in terms of consumer surplus, of the four alternative divestiture options under consideration.

Proposition 4. *The induced change in consumer surplus is more favorable as we move from Option 1 (proportional divestiture) to Option 2 (switch from voting stock to preferred stock), irrespective of being a complete or partial divestiture, if and only if*

$$k_{AA}\left(w_{B0}^1 - w_{B0}^2 \frac{k_{BB} + S_{BB}^1}{k_{BB} + S_{BB}^2}\right) + S_{BA}^1 > 0.$$

∎

When comparing Option 1 and Option 2, three differences are worth stressing. First, after a partial divestiture, firm A will retain some influence over firm B's actions, i.e. firm B will still place some control weight on firm A's payoff. However, for the same divestiture of voting rights, this remaining weight will be larger under Option 2: $w_{B0}^1 < w_{B0}^2$. The proportional divestiture removes some voting shares and distributes them to new shareholders, while Option 2 merely eliminates the voting rights. Given the creation of new shareholders, this reduces the relative control weight of firm A on firm B more in the case of Option 1. The lower w_{B0}^1 and the larger w_{B0}^2 the more likely it is that Option 1 is better for consumers. Second, the effect on k_{BB} also differs across the two options. In fact, the original shareholders with control and financial interests in firm B will also be differently affected by the two options. For the same reason as above, Option 2 leads to larger relative weights because there are no new shareholders involved. The increase in correlation between control and financial interest makes firm B's decisions more independent from firm A under this option. This independence is beneficial for consumers but, considering these two effects combined, the first term in the condition in Proposition 1 can be shown to be negative. Finally, Option 1 creates common shareholders, i.e. shareholders with control of firm B who care for firm A's profit, a factor that favors Option 2, in terms of its impact on consumers. This is represented by the term S_{BA}^1 in the condition above. In case of a complete divestiture, we have $w_{B0}^1 = w_{B0}^2 = 0$ and the condition above always holds (in the absence of common shareholders). Note, in particular, that if financial interest

---

[14] In rigor, when some of firm A's voting stock in firm B is sold to infinitesimal shareholders, there is a scaling effect that changes both k_{BB} and v_{B0} in the same proportion. We omit this in the discussion because it has no effect on γ_B.

in firm A is very diluted, which implies both k_{AA} and S^1_{BA} close to zero, the proportional divestiture will lead to the distribution of voting shares in firm B to a very high number of very small shareholders and, as a result, the two options would tend to be equivalent in terms of their induced change in consumer surplus.

Further, if $f(v_{Bn}) = v_{Bn}$ and $\sum_{n=0}^{N} v_{Bn} = \sum_{n=1}^{N} t_{An} = 1$, the condition in Proposition 4 becomes $\alpha v_{B0} \sum_{n=1}^{N} t_{An} t_{An} > 0$, which always holds. With this particular weight structure, turning voting shares into preferred stock is better than distributing shares proportionally both in the case of a complete and in the case of a partial divestiture (Option 2 is better than Option 1) because the induced effects on γ_B are different.[15] The only effect associated with the switch from voting stock to preferred stock (Option 2) is a reduction in v_{B0} while the proportional divestiture (Option 1) leads to increases k_{BB}, k_{BA} and k_{AB}. The two latter effects correspond to the creation of common shareholders, a situation that we describe in detail in Section 5. The dominant effect that makes the proportional divestiture perform worse in terms of impact on consumer surplus is then that this proportional divestiture creates shareholders with (direct) voting shares in firm B and financial interest in firm A. In fact, the higher the financial interest a shareholder has in firm A, the larger the percentage of voting shares that this shareholder will get with the proportional divestiture of voting stock initially owned by firm A, and this will lead to an increase in the weight firm B places in firm A's profit.

Proposition 5. *Consider the case of a partial divestiture. The induced change in consumer surplus is more favorable as we move from Option 3 (sale to small shareholders) to Option 4 (sale to one large shareholder). In the case of a complete divestiture both options are equivalent in terms of the induced change in consumer surplus.∎*

The intuition that Option 4 (sale to one large shareholder) is better than Option 3 (sale to small shareholders) in the case of a partial divestiture is akin to part of Lemma 2′, where we state that γ_B is decreasing in k_{BB}: the greater the concentration of firm B's shareholdings, the less weight firm B places on firm A's profits. In the limit, when $k_{BB} \to 0$, firm A's voting share in firm B grants firm A effective control over firm B, that is, firm B maximizes firm A's profits (assuming that firm A voting shareholders also have a financial interest in firm A). The difference between Option 3 and Option 4 is that k_{BB} remains constant under the former, whereas k_{BB} increases in the latter which, by Lemma 2′, implies a lower γ_B and a higher consumer surplus.

In other words, the contrast between the sale to small shareholders and the sale to one large shareholder corresponds to the *countervailing effect* of shareholder concentration: strong independent shareholders in firm B have the beneficial effect of counterweighting the negative effect (from a consumer surplus perspective) of firm A's partial ownership of firm B. There is an interesting analogy with the countervailing effect of buyer power in vertical relations (e.g., Dobson and Waterson, 1997). In general, market share concentration is bad for consumer welfare; but to the extent that there already is market power at one level of the value chain, an increase in concentration at a different level (e.g., downstream) may be welfare-enhancing.

While we show that a partial sale to one large shareholder is better than the sale to an infinite number of infinitesimal shareholders, these are not the only possibilities to consider. In fact, the highest consumer surplus corresponds to a sale pattern that maximizes the post-divestiture value of k_{BB}. If there is a perfect correlation between control and financial interest in firm B, this would correspond to selling v_{B0} to the largest independent shareholder in firm B.

It should be highlighted that Options 3 and 4 are equivalent in terms of their impact on consumer surplus in the case of a complete

divestiture of voting stock. Given the inexistence of common shareholders, such divestiture completely separates both firms with the exception of some remaining financial interest of firm A in firm B. Hence, after the divestiture, $\gamma_A = s_{B0}(1 - V_B)$ and $\gamma_B = 0$.

Proposition 6. *Consider the case of a complete divestiture. Then, the induced change in consumer surplus is more favorable: (i) as we move from Option 2 (switch from voting stock to preferred stock) to Option 3 (sale to small shareholders) or to Option 4 (sale to one large shareholder); (ii) as we move from Option 1 (proportional divestiture) to Option 3 (sale to small shareholders) or to Option 4 (sale to one large shareholder).∎*

To understand why a complete sale to infinitesimal shareholders is better than turning voting stock into preferred stock (Option 3 is better than Option 2), it may help to think of a complete sale of v_{B0} to an infinite number of infinitesimal shareholders (Option 3) as a two-step process: first v_{B0} is turned into preferred stock (Option 2); and then this preferred stock is distributed to an infinite number of infinitesimal shareholders. Strictly speaking, the two steps lead to a different final arrangement than Option 3. However, to the extent that the stock is distributed to infinitesimal shareholders, it does not matter whether it is voting or preferred stock.

The crucial point is then to understand the impact of the second step above: transferring preferred stock from firm A to an infinite number of infinitesimal shareholders. From Eq. (8), a decrease in t_{B0} leads to a decrease in γ_A. This makes sense: to the extent that firm A reduces its financial interest in firm B, it will place a lower weight on firm B's profit. Having established that the transfer of t_{B0} leads to a lower γ_A, we must now add that it has zero effect on γ_B. This may appear to contradict Lemma 2′, where we stated that a decrease in t_{B0} leads to an increase in γ_B; but in fact such result requires $w_{B0} > 0$. Since the first step above leads to $v_{B0} = w_{B0} = 0$, the subsequent transfer of preferred stock has no effect on γ_B. We thus conclude that the second step leads to a further decrease in γ_A, which by Lemma 1 corresponds to an increase in consumer surplus, and which finally proves that the complete sale of v_{B0} leads to a greater increase in consumer surplus than the switch from voting stock to preferred stock.

A complete proportional divestiture is not as good as the sale to infinitesimal shareholders essentially because it creates common shareholders. In the limit case in which firm A's shareholders were infinitesimal, the two divestitures would have the same effect.

4.1. Transaction costs

As mentioned at the beginning of the section, our results — and in particular Proposition 6 — do not take into account the transaction costs involved with each of the options. Turning voting stock into preferred stock, or transferring the ownership from firm A to firm A's shareholders, does not require valuing the shares — and does not require any financial transaction to take place. By contrast, selling shares to a third party creates a host of potential problems. For example, if the time frame for the sale is too narrow then there may not be enough market demand; whereas if the time frame is too wide then the beneficial effects of divestiture may take too long to take place; and for a given time frame, potential buyers may delay strategically their purchases. Moreover, if the Competition Authority is to determine the price at which the shares are to be sold, then there is the obvious problem of share valuation.

For all these reasons, our results must be taken with a grain of salt: While Options 3 and 4 are better than Option 2 in terms of consumer surplus, the transactions costs they imply may warrant the simpler option of simply changing the nature of firm A's shares in firm B.

5. Extension: common shareholders

The results in the previous section were based on the assumption that, initially, no shareholder owns shares of both firms. In some cases

---

[15] Both options leave γ_A unchanged. With Option 2, γ_A does not change by definition. As for Option 1, γ_A is not affected because indirect financial interest is transformed, in the same proportion, in direct financial interest.

(including the Portugal Telecom case described in the next section) there are indeed common shareholders. How does this change the analysis? In this section, we show that most results remain qualitatively the same. However, as Lemma 2 and 2′ illustrate, there are additional effects to consider, on the one hand and the math becomes considerably more complex, on the other hand.

For each of the Propositions in the previous section, the relevant condition with common shareholders is presented in the corresponding proofs in Appendix A. In what follows, we discuss the main implications of introducing common shareholders, for each of the divestiture options under analysis.

With common shareholders, Option 1, the proportional divestiture, increases consumer welfare if and only if γ_B decreases,[16] which occurs when

$$K_{BB}\left(S_{BA} + k_{AA}\Delta w_{B0}^1\right) < K_{BA}\left(S_{BB} + k_{AB}\Delta w_{B0}^1\right).$$

In other words, whether the proportional divestiture is good or bad for consumers depends on the interplay between four terms: (1) K_{BB}, (2) $(S_{BA} + k_{AA}\Delta w_{B0}^1)$, (3) K_{BA}, and (4) $(S_{BB} + k_{AB}\Delta w_{B0}^1)$.

Note that the first and third terms are positive, whereas the second and fourth terms can be either positive or negative.

The first term measures the extent to which, at the outset, the ownership of voting stock in firm B is correlated with financial interest in the same firm, i.e., it measures the extent to which shareholders with the ability to influence firm B's decisions are interested in firm B's profit.

The second term, which can be written as $\sum_{n=1}^{N}(\Delta w_{Bn}^1 + w_{An}\Delta w_{B0}^1)t_{An}$, is positive if those shareholders who will have an increase in their direct control weight larger than the decrease in the indirect control weight in firm B, also have a large t_{An}.[17]

The third term, is similar to the first one and measures the correlation between shareholders initial direct and indirect voting rights in firm B and their financial interest in firm A, and thus captures to which extent voters in firm B are interested in firm A's profit.

Finally, the fourth term, $\sum_{n=1}^{N}(\Delta w_{Bn}^1 + w_{An}\Delta w_{B0}^1)t_{Bn}$, is positive if those shareholders who will have an increase in their direct control weight larger than the decrease in indirect control weight in firm B also have a high financial interest in firm B, t_{Bn}.

If the second term is positive and the fourth term is negative, then the condition for consumer welfare to increase is never verified. This happens because if the shareholders to whom firm B will attribute a larger weight care a lot for firm A's profit and a little for firm B's profit, then they will instruct firm B to give a higher weight to firm A's profit. As a result, γ_B will increase and consumers will be worse off.

If the second term is negative and the fourth term is positive, then exactly the opposite effects take place and the condition for consumer welfare to increase is trivially satisfied.

If the second and fourth terms have the same sign, say, are both positive, then the condition for consumer welfare to increase depends on their magnitude and also on the magnitude of the first and third terms. Note, however, that if, at the outset, direct and indirect voters in firm B cared more for firm A's profits than for firm B's profit, i.e., if, at the outset, the first term is smaller than the third one, then it is less likely that the divestiture will impact consumers negatively because firm B already placed a large weight on firm A's profit at the outset.

As for the impact of the existence of common shareholders on Option 2 (removing the voting rights associated with v_{B0}), recall that such operation increases the degree of control by firm B's shareholders. In particular, by merely eliminating control rights (previously owned by firm A), this option leverages up the control power which is exercised by private shareholders. This in turn leads to an increase in k_{BA}, an effect

not considered in our previous analysis (k_{AB} in turn remains constant, whereas k_{BB} increases and k_{AA} remains constant, as considered before). To the extent that $k_{AA}k_{BB} > k_{BA}k_{AB}$, the effect on γ_B of this divestiture is still negative as before. However, the effect turns out to be lower in absolute value.

Finally, it should be noted that when common shareholders exist the complete sale option (for instance, Option 3, the sale to an infinite number of small shareholders) may no longer have a positive effect on consumer surplus. In fact, although, as before, γ_A decreases inequivocally (because of the decrease in t_{B0}) the change in γ_B has the same sign of

$$-w_{B0}(k_{AA}k_{BB} - k_{AB}k_{BA}) + S_{BA}^3(k_{BB} + w_{B0}k_{AB}) - S_{BB}^3(k_{BA} + w_{B0}k_{AA}) + \\ + V_B v_{B0}\left(S_{BA}^3 + k_{BA}\right)(k_{BA} + w_{B0}k_{AA})$$

which has an uncertain sign, if $k_{BA}^3 = k_{BA} + S_{BA}^3$ is positive (otherwise, it is negative if $k_{AA}k_{BB} > k_{BA}k_{AB}$). The reason is, as before, that such sale reduces t_{B0} and v_{B0}, whose induced effects on γ_B go in opposite directions. A positive k_{BA}^3 means that firm B will still care for firm A's profit even if firm A loses its control shares in firm B. As the loss in control implies a loss in financial interest, firm A's aggregate profits, which accrue to shareholders who have control in firm B, will include a lower percentage of firm B's profit. Hence, firm B will place a larger weight on firm A's profit and a lower weight on its own profit.

6. Application: PT and PTM

As the starting paragraph in the paper suggests, the questions we examine are not of pure intellectual interest; they also correspond to actual situations where concrete regulatory decisions need to be made. In this section, we consider a retrospective application of our framework: the divestiture of Portugal Telecom's, Portugal's largest telecommunications operator, holdings in PTM. Until November 2007 Portugal Telecom (PT) held a 58% share of PT Multimedia (PTM). The two firms operated in several markets as the two main "competitors" (sometimes the sole competitors). Under pressure from the Portuguese government, PT agreed to divest its shares in PTM. PT management's proposal was then to divest its share in PTM to PT's shareholders, in proportion to their holdings.

Table 1 lists the main shareholders in PT and PTM before the divestiture. As can be seen, there were a few relatively large shareholders. Moreover, aside from the fact that PT owned a share in PTM, there was a significant overlap in ownership of both firms. In fact, three of the four larger shareholders in PTM, excluding PT, also owned a

Table 1
Large shareholders in PT and PTM (as of May 2006).

Shareholder	Shares		Voting rights	
	PT	PTM	PT	PTM
PT	–	58.43%	–	10.00%
PTM	0.00%	–	0.00%	–
Telefónica	9.96%	0.00%	9.96%	0.00%
Grupo Espírito Santo	7.77%	6.96%	7.77%	6.96%
Brandes Investments Partners	7.41%	0.00%	7.41%	0.00%
Ongoing Strategy Investments	5.35%	0.00%	5.35%	0.00%
Grupo Caixa Geral de Depósitos	5.11%	11.26%	5.11%	10.00%
Telmex	3.41%	0.00%	3.41%	0.00%
Paulson Co. Inc.	0.00%	2.34%	0.00%	2.34%
Merrill Lynch International	2.20%	0.00%	2.20%	0.00%
Fidelity	2.09%	0.00%	2.09%	0.00%
Grupo Barclays	2.06%	0.00%	2.06%	0.00%
Capital Group Companies	2.04%	0.00%	2.04%	0.00%
Grupo Visabeira	2.01%	0.00%	2.01%	0.00%
Controlinveste/Joaquim Oliveira	2.00%	3.77%	2.00%	3.77%
Grupo BPI	0.00%	5.16%	0.00%	5.16%
Cofina. SGPS. S.A.	0.00%	2.23%	0.00%	2.23%
Total	53.75%	29.38%	53.75%	28.12%

---

[16] Recall that γ_A is not affected by a proportional divestiture (see Proposition 1).

[17] Note that the change in direct control weight is given by Δw_{Bn}^1 which is mitigated by the reduction in indirect control weight, $w_{An}\Delta w_{B0}^1 < 0$.
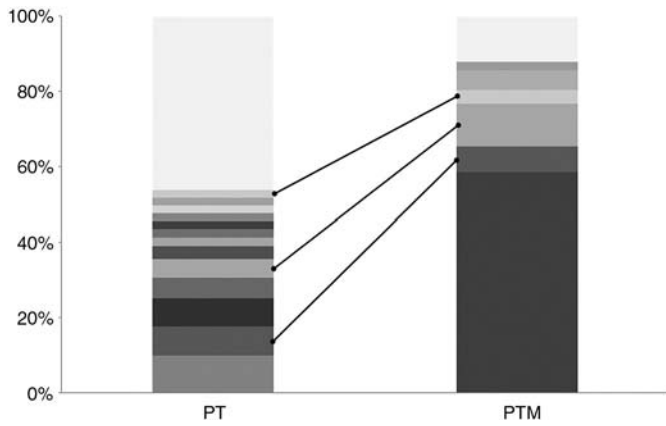
Fig. 2. PT and PTM shareholder structure and corresponding common shareholders.

significant percentage of shares in PT. Fig. 2 presents the shareholder structure of both firms and identifies the shareholders present in both firms at the time.

In Section 2, we assumed the weight attributed by each firm (in its objective function) to a given shareholder is given by a function of the shareholder's percentage of voting shares, that is, v_{in}. Now, an important remark that should be made at this point is that, according to PTM's statutes at the time, voting rights were capped at 10%. This means that a shareholder owning more than 10% of PTM's shares was only entitled to 10% of the votes.

In what follows we consider as alternatives the four options considered in the previous section, namely:

1. Transfer of PT's share in PTM to PT's shareholders in proportion to their initial shares in PT. This was the option proposed by PT and actually implemented.

2. Change of PT's share in PTM from voting stock to preferred stock.

Table 2
Effect of various forms of divestiture, assuming no common shareholders before the divestiture, with $f(v_{in}) = (v_{in})^z$.

	Base case	Option 1	Option 2	Option 3	Option 4
$z = 1$					
k_{AA}	0.058	0.058	0.058	0.058	0.058
k_{AB}	0.000	0.034	0.000	0.000	0.000
k_{BA}	0.000	0.031	0.000	0.000	0.000
k_{BB}	0.054	0.053	0.074	0.074	0.208
w_{B0}	0.262	0.000	0.000	0.000	0.000
t_{B0}	0.584	0.000	0.584	0.000	0.000
γ_A	0.584	0.584	0.584	0.000	0.000
γ_B	0.241	0.582	0.000	0.000	0.000
$z = 2$					
k_{AA}	0.072	0.072	0.072	0.072	0.072
k_{AB}	0.000	0.042	0.000	0.000	0.000
k_{BA}	0.000	0.026	0.000	0.000	0.000
k_{BB}	0.057	0.070	0.086	0.086	0.255
w_{B0}	0.340	0.000	0.000	0.000	0.000
t_{B0}	0.584	0.000	0.584	0.000	0.000
γ_A	0.584	0.584	0.584	0.000	0.000
γ_B	0.346	0.364	0.000	0.000	0.000
$z = 3$					
k_{AA}	0.081	0.081	0.081	0.081	0.081
k_{AB}	0.000	0.047	0.000	0.000	0.000
k_{BA}	0.000	0.018	0.000	0.000	0.000
k_{BB}	0.057	0.084	0.094	0.094	0.287
w_{B0}	0.394	0.000	0.000	0.000	0.000
t_{B0}	0.584	0.000	0.584	0.000	0.000
γ_A	0.584	0.584	0.584	0.000	0.000
γ_B	0.420	0.219	0.000	0.000	0.000

Table 3
Effect of various forms of divestiture, assuming common shareholders before the divestiture, with $f(v_{in}) = (v_{in})^z$.

	Base Case	Option 1	Option 2	Option 3	Option 4
$z = 1$					
k_{AA}	0.058	0.058	0.058	0.058	0.058
k_{AB}	0.022	0.056	0.022	0.022	0.022
k_{BA}	0.030	0.049	0.040	0.040	0.030
k_{BB}	0.054	0.071	0.074	0.074	0.208
w_{B0}	0.262	0.000	0.000	0.000	0.000
t_{B0}	0.584	0.000	0.584	0.000	0.000
γ_A	0.966	0.966	0.966	0.382	0.382
γ_B	0.519	0.689	0.413	0.545	0.142
$z = 2$					
k_{AA}	0.072	0.072	0.072	0.072	0.072
k_{AB}	0.023	0.066	0.023	0.023	0.023
k_{BA}	0.031	0.057	0.047	0.047	0.031
k_{BB}	0.057	0.096	0.086	0.086	0.255
w_{B0}	0.340	0.000	0.000	0.000	0.000
t_{B0}	0.584	0.000	0.584	0.000	0.000
γ_A	0.908	0.908	0.908	0.323	0.323
γ_B	0.574	0.599	0.416	0.550	0.122
$z = 3$					
k_{AA}	0.081	0.081	0.081	0.081	0.081
k_{AB}	0.021	0.069	0.021	0.021	0.021
k_{BA}	0.031	0.061	0.051	0.051	0.031
k_{BB}	0.057	0.110	0.094	0.094	0.287
w_{B0}	0.394	0.000	0.000	0.000	0.000
t_{B0}	0.584	0.000	0.584	0.000	0.000
γ_A	0.847	0.847	0.847	0.262	0.262
γ_B	0.613	0.554	0.410	0.539	0.107

3. Sale of PT's share in PTM to small shareholders different from the current shareholders.

4. Sale of PT's share in PTM to a new large shareholder.

In this application, we assume $f(v_{in}) = (v_{in})^z$ and consider three alternative arbitrary values for z : $z = 1$; $z = 2$ and $z = 3$. Further, we denote PT by firm A and PTM by firm B. In all cases, PTM holds no share in PT, that is, $t_{A0} = 0$. Our focus is on the value of t_{B0}, positive in the initial situation and zero in all other scenarios. Specifically, we proceed as follows. For each scenario, we compute the values of t_{B0}, v_{B0}, k_{AA}, k_{BB}, k_{AB} and k_{BA}. These values, together with the corresponding γ_A and γ_B for the three alternative values of z, are presented in Tables 2 and 3. Table 2 assumes initially that there are no common shareholders, while Table 3 uses the shareholder structure presented in Table 1.

As most of PT's shareholders would own a share in PTM below 10% after the proportional divestiture one can consider the proposed operation (Option 1) as a two-step operation. In a first step, the legal caps on voting rights are removed and, in a second step, the shares are distributed to PT's shareholders in proportion to their initial shares in PT. The first step corresponds to turning some non-voting shares (those corresponding to holdings in excess of 10%) into voting shares. From Proposition 2, this has a negative effect on consumer welfare. The second step is neutral in terms of consumer welfare when $z = 1$. As discussed after Proposition 1, if there is only one type of shares and if shareholder weights are equal to their percentage of voting stock, the proportional divestiture merely turns indirect holdings into direct ones.[18] Hence, if a shareholder's weight is given by its voting rights, then PT's proposal (Option 1) actually decreases consumer welfare, provided that z is not much higher than 1. This can be seen at the bottom of

---

[18] To be rigorous, the legal caps are not completely removed. This happens because one of the shareholders exceeds the 10% cap before the divestiture and another one exceeds it afterwards. The percentage of shares above 10% corresponds to only 5.75%. Therefore, the ensuing proportional distribution of shares is not completely neutral. The difference, however, is quite small. Moreover, proposition 1 is only neutral if $\sum_{n=0}^{N} v_{Bn} = \sum_{n=1}^{N} t_{An} = 1$, which does not hold in this example due to the inexistence of information about small shareholders.

each panel in Table 2, where the estimated values for the γ's are reported. The reason for this is that, because of legal caps on voting rights, PT's voting rights in PTM (in the initial situation) are considerably lower than its shareholdings. Now, by transferring PT's shares in PTM to PT's large shareholders, PT is actually increasing the concentration of voting rights in PTM, since the shares it transfers correspond to greater voting rights than in the initial situation. In other words, by relinquishing 1% of voting rights, PT is increasing PT's shareholders' voting rights by more than 1%. However, when z is sufficiently large (in Table 2, when z = 3), Option 1 may lead to a reduction in γ_B. The reason for this is that with a sufficiently convex $f(.)$ function, the distribution of voting stock among its own shareholders leads to a reduction in the (aggregate) weight given by firm B (PTM) to shareholders with financial interests in firm A (PT). Table 2 also illustrates the findings of Propositions 3 and 4. Without common shareholders, any of the sale options completely separates the firms and brings both γ_A and γ_B down to zero. Finally, Option 2 has no effect on γ_A and also brings γ_B to zero. In fact, in the absence of common shareholders, the removal of firm B voting rights in firm A (either by sale or simply by converting this shares into preferred stock) makes firm B completely immune to the influence of any shareholders with financial interests in firm A and, hence, firm B will give no weight to its competitor, i.e. γ_B becomes equal to zero in Options 2, 3 and 4.

Table 3 presents the same information but with common shareholders. The main qualitative difference is the fact that the sale to infinitesimal shareholders may have a negative effect on consumer surplus, as the two weights, γ_A and γ_B, move in opposite directions, as mentioned at the end of Section 5. A complete sale reduces both v_{B0} and t_{B0} to zero and ends any (indirect) control firm A's shareholders might have in firm B. However, given that some shareholders own stock in both firms, there will subsist shareholders with some (direct) control over firm B that also have a financial interest in firm A and that now care less for firm B's operational profit. Recall that after the divestiture, firm A's aggregate profit no longer depends on firm B's operational profit because the sale put an end to firm A's financial interest in firm B. Hence, these shareholders will instruct firm B to attribute a higher weight to firm A's profit.

The remaining alternatives confirm the expectations from Propositions 4, 5 and 6, which were obtained for the case of no common shareholders and therefore do not directly apply to Table 3. Namely that Option 4 is better than Option 3, and that Option 2 is better than Option 1 (the option actually implemented). As mentioned above, we should qualify this assertion by recognizing that the transactions costs of each option might differ. In particular, one advantage of Options 1 and 2 is that they do not require a market sale, with all the transactions costs this may imply.

7. Conclusion

We provided a series of necessary and sufficient conditions that allow us to rank alternative options for divestiture of firm A's holdings in firm B. Overall, three robust ideas stand out from our results: First, a participation that induces control is more damaging to consumer welfare than a passive participation (though both decrease consumer surplus). Second, the "proportional" method (which has been used in recent divestiture arrangements) generally performs worse than turning voting stock into preferred stock or selling shares to a third party. Third, the concentration of control among independent shareholders in the target firm is beneficial from a consumer surplus point of view (the "countervailing" effect).

Appendix A

Proof of Lemma 1. Firm i's first-order condition is given by

$$f\left(p_i, p_j; \gamma_i, \gamma_j\right) = \frac{\partial \pi_i}{\partial p_i} + \gamma_i \frac{\partial \pi_j}{\partial p_i} = 0.$$

Since $\partial \pi_j / \partial p_i > 0$, it follows that $\partial f(p_i, p_j; \gamma_i, \gamma_j)/\partial \gamma_i > 0$. Since moreover $\partial^2 \omega_i / \partial p_i \partial p_j = \partial^2 \pi_i / \partial p_i \partial p_j + \gamma_i \partial^2 \pi_j / \partial p_i \partial p_j > 0$, standard supermodularity results (e.g., Theorems 2.3 in Vives, 2000) imply that equilibrium prices are increasing in γ_i. Finally, since consumer surplus is decreasing in prices the result follows.■

Proof of Lemma 2. Taking derivatives and simplifying, we get

$$\frac{\partial \gamma_B}{\partial K_{BA}} = \frac{K_{BB}}{\left(K_{BB} + K_{BA} t_{B0}\right)^2} > 0$$

$$\frac{\partial \gamma_B}{\partial K_{BB}} = -\frac{K_{BA}}{\left(K_{BB} + K_{BA} t_{B0}\right)^2} < 0$$

$$\frac{\partial \gamma_B}{\partial t_{B0}} = -\frac{K_{BA}^2}{\left(K_{BB} + K_{BA} t_{B0}\right)^2} < 0$$

$$\frac{\partial \gamma_A}{\partial t_{B0}} = 1 > 0$$

$$\frac{\partial \gamma_A}{\partial K_{AB}} = \frac{1}{K_{AA}} > 0$$

$$\frac{\partial \gamma_A}{\partial K_{AA}} = -\frac{K_{AB}}{K_{AA}^2} \leq 0.$$

■

Proof of Lemma 2′. Taking derivatives and simplifying, we get

$$\frac{\partial \gamma_B}{\partial t_{B0}} = -\frac{\left(k_{BA} + w_{B0} k_{AA}\right)^2}{\left(k_{BB} + w_{B0}(k_{AB} + t_{B0} k_{AA}) + t_{B0} k_{BA}\right)^2} < 0$$

$$\frac{\partial \gamma_B}{\partial v_{B0}} = \frac{\left(k_{AA} k_{BB} - k_{AB} k_{BA}\right)}{\left(k_{BB} + w_{B0}(k_{AB} + t_{B0} k_{AA}) + t_{B0} k_{BA}\right)^2} \frac{\partial w_{B0}}{\partial v_{B0}}$$

$$\frac{\partial \gamma_B}{\partial k_{AA}} = w_{B0} \frac{\partial \gamma_B}{\partial k_{BA}} = \frac{\left(k_{BB} + w_{B0} k_{AB}\right) w_{B0}}{\left(k_{BB} + w_{B0}(k_{AB} + t_{B0} k_{AA}) + t_{B0} k_{BA}\right)^2} > 0$$

$$\frac{\partial \gamma_B}{\partial k_{BB}} = \frac{1}{w_{B0}} \frac{\partial \gamma_B}{\partial k_{AB}} = -\frac{\left(k_{BA} + w_{B0} k_{AA}\right)}{\left(k_{BB} + w_{B0}(k_{AB} + t_{B0} k_{AA}) + t_{B0} k_{BA}\right)^2} < 0$$

and

$$\frac{\partial \gamma_A}{\partial t_{B0}} = 1 > 0$$

$$\frac{\partial \gamma_A}{\partial w_{B0}} = \frac{1}{k_{AA}} > 0$$

$$\frac{\partial \gamma_A}{\partial k_{AA}} = -\frac{k_{AB}}{k_{AA}^2} \leq 0.$$

■

In the proofs of all propositions, $o = 0,1,2,3,4$ denotes the value taken by the relevant variables after divestiture o took place. The superscript 0 referring to the baseline case is omitted. Additionally, $\Delta x^o := x^o - x$. In all proofs, we start with the most complex case (general weight function for w_{in}, with common shareholders), then present the case of the general weight function for w_{in}, without common shareholders and, finally, the corresponding expression for the case of $f(v_{in}) = v_{in}$.

It should be noted that under all divestiture options considered, there are no changes in the distribution of firm A's shares, so $t_{An}^o = t_{An}$, $v_{An}^o = v_{An}$ and $w_{An}^o = w_{An}$, for $o=1,2,3,4$. Hence, $K_{AA}^o = k_{AA}^o = k_{AA}$. With the exception of the proportional divestiture (Option 1) there are also no changes in t_{Bn} for the original shareholders. Hence, $K_{AB}^o = k_{AB}^o = k_{AB}$ for $o = 2,3,4$.

Given Lemma 1, the effects of the different divestiture options on consumer surplus depend on their effects on the weights each firm

places on its competitor. Under Option $o = 0,1,...,4$, the weights are given by

$$\gamma_A^o = t_{B0}^o + \frac{K_{AB}^o}{K_{AA}^o}. \tag{9}$$

$$\gamma_B^o = \frac{K_{BA}^o}{K_{BB}^o + K_{BA}^o t_{B0}^o}. \tag{10}$$

and the variation in the weights as one moves from Option o' to Option o is given by:

$$\gamma_A^o - \gamma_A^{o'} = t_{B0}^o - t_{B0}^{o'} + \frac{K_{AB}^o - K_{AB}^{o'}}{K_{AA}} \tag{11}$$

$$\gamma_B^o - \gamma_B^{o'} = \frac{K_{BA}^o}{K_{BB}^o + K_{BA}^o t_{B0}^o} - \frac{K_{BA}^{o'}}{K_{BB}^{o'} + K_{BA}^{o'} t_{B0}^{o'}}. \tag{12}$$

In the proofs below, we only need to know the sign of $\gamma_A^o - \gamma_A^{o'}$ and $\gamma_B^o - \gamma_B^{o'}$. The sign of the latter is given by

$$sign\left(\gamma_B^o - \gamma_B^{o'}\right) = sign\left(K_{BB}^{o'}K_{BA}^o - K_{BB}^o K_{BA}^{o'} + K_{BA}^o K_{BA}^{o'}\left(t_{B0}^{o'} - t_{B0}^o\right)\right). \tag{13}$$

Proof of Proposition 1. Assume a fraction $\alpha, \alpha \leq 1$, is being divested of both v_{B0} and t_{B0}. With the divestiture, shares are distributed among the shareholders of the parent company, firm A, in proportion to their holdings, t_{An}, and we have:

$$t_{Bn}^1 = t_{Bn} + \alpha t_{B0} t_{An}$$
$$v_{Bn}^1 = v_{Bn} + \alpha v_{B0} t_{An}.$$

As a result of these changes there will be a variation in the weights given by firm B to each private shareholder $n = 1, ..., N$. This variation is given by

$$\Delta w_{Bn}^1 = w_{Bn}^1 - w_{Bn}$$
$$= \frac{f(v_{Bn} + \alpha v_{B0} t_{An})}{\sum_{n=1}^{N} f(v_{Bn} + \alpha v_{B0} t_{An}) + f((1-\alpha)v_{B0})} - \frac{f(v_{Bn})}{\sum_{n=1}^{N} f(v_{Bn}) + f(v_{B0})}.$$

Note that for some shareholders this variation may be positive while, for others, it may be negative. In general, the denominator of w_{Bn} may either increase or decrease and the numerator increases only for those shareholders with $t_{An} > 0$.

As for firm A:

$$\Delta w_{B0}^1 = w_{B0}^1 - w_{B0}$$
$$= \frac{f((1-\alpha)v_{B0})}{\sum_{n=1}^{N} f(v_{Bn} + \alpha v_{B0} t_{An}) + f((1-\alpha)v_{B0})} - \frac{f(v_{B0})}{\sum_{n=1}^{N} f(v_{Bn}) + f(v_{B0})} < 0.$$

After the divestiture, we have:

$$k_{AB}^1 = \sum_{n=1}^{N} w_{An}^1 t_{Bn}^1 = \sum_{n=1}^{N} w_{An}(t_{Bn} + \alpha t_{B0} t_{An}) = k_{AB} + \alpha t_{B0} k_{AA}$$

$$k_{BA}^1 = \sum_{n=1}^{N} w_{Bn}^1 t_{An}^1 = \sum_{n=1}^{N} \left(w_{Bn} + \Delta w_{Bn}^1\right) t_{An} = k_{BA} + \sum_{n=1}^{N} \Delta w_{Bn}^1 t_{An} = k_{BA} + S_{BA}^1$$

$$k_{BB}^1 = \sum_{n=1}^{N} w_{Bn}^1 t_{Bn}^1 = \sum_{n=1}^{N} \left(w_{Bn} + \Delta w_{Bn}^1\right)(t_{Bn} + \alpha t_{B0} t_{An}) = k_{BB} + S_{BB}^1 + \alpha t_{B0}\left(k_{BA} + S_{BA}^1\right)$$

where

$$S_{ij}^o = \sum_{n=1}^{N} \Delta w_{in}^o t_{jn}$$

Hence,

$$K_{AB}^1 = k_{AB}^1 = k_{AB} + \alpha t_{B0} k_{AA} = K_{AB} + \alpha t_{B0} k_{AA}$$

$$K_{BA}^1 = k_{BA}^1 + w_{B0}^1 k_{AA}^1 = k_{BA} + S_{BA}^1 + w_{B0} k_{AA} + \Delta w_{B0}^1 k_{AA} = K_{BA} + S_{BA}^1 + \Delta w_{B0}^1 k_{AA}$$

$$K_{BB}^1 = k_{BB}^1 + w_{B0}^1 k_{AB}^1 = k_{BB} + S_{BB}^1 + \alpha t_{B0}\left(k_{BA} + S_{BA}^1\right) + w_{B0}^1\left(k_{AB} + \alpha t_{B0} k_{AA}\right)$$

$$= K_{BB} + \alpha t_{B0}\left(K_{BA} + S_{BA}^1 + \Delta w_{B0}^1 k_{AA}\right) + S_{BB}^1 + \Delta w_{B0}^1 k_{AB}.$$

Inserting these expressions into Eqs. (11) and (13) we obtain:

$$\gamma_A^1 - \gamma_A = t_{B0}^1 - t_{B0} + \frac{K_{AB}^1 - K_{AB}}{K_{AA}} = (1-\alpha)t_{B0} - t_{B0} + \frac{K_{AB} + \alpha t_{B0} K_{AA} - K_{AB}}{K_{AA}} = 0$$

$$sign\left(\gamma_B^1 - \gamma_B\right) = sign\left(K_{BB}\left(S_{BA}^1 + \Delta w_{B0}^1 k_{AA}\right) - K_{BA}\left(S_{BB}^1 + \Delta w_{B0}^1 k_{AB}\right)\right).$$

Therefore, Option 1 does not lead to any change in γ_A and it leads to a reduction in γ_B if and only if

$$K_{BB}\left(S_{BA}^1 + \Delta w_{B0}^1 k_{AA}\right) - K_{BA}\left(S_{BB}^1 + \Delta w_{B0}^1 k_{AB}\right) < 0. \tag{14}$$

With no-common shareholders we have $k_{AB} = k_{BA} = 0$, $K_{BB} = k_{BB}$ and $K_{BA} = w_{B0}k_{AA}$. Denote by N_i the original set of firm i's shareholders, with $N_A \cap N_B = \varnothing$. For firm A's original shareholders, i.e. for $n \in N_A$, we have:

$$\Delta w_{Bn}^1 = \frac{f(\alpha v_{B0} t_{An})}{\sum_{n \in N_A} f(\alpha v_{B0} t_{An}) + \sum_{n \in N_B} f(v_{Bn}) + f((1-\alpha)v_{B0})} - 0 > 0,$$

from where we have $S_{BA}^1 > 0$.

As for firm B's original shareholders, i.e. for $n \in N_B$, we have

$$\Delta w_{Bn}^1 = \frac{f(v_{Bn})}{\sum_{n \in N_A} f(\alpha v_{B0} t_{An}) + \sum_{n \in N_B} f(v_{Bn}) + f((1-\alpha)v_{B0})}$$
$$- \frac{f(v_{Bn})}{\sum_{n=1}^{N} f(v_{Bn}) + f(v_{B0})} > 0,$$

which implies $S_{BB}^1 > 0$.

In case of no common shareholders, Eq. (14) becomes

$$k_{BB}\left(S_{BA}^1 + \Delta w_{B0}^1 k_{AA}\right) - w_{B0} k_{AA} S_{BB}^1 < 0. \tag{15}$$

Finally, if $f(v_{in}) = v_{in}$ and $\sum_{n=0}^{N} v_{Bn} = \sum_{n=1}^{N} t_{An} = 1$, we have

$$\Delta w_{B0}^1 = \frac{(1-\alpha)v_{B0}}{\sum_{n=1}^{N}(v_{Bn} + \alpha v_{B0} t_{An}) + (1-\alpha)v_{B0}} - \frac{v_{B0}}{\sum_{n=1}^{N} v_{Bn} + v_{B0}} = -\alpha v_{B0}$$

$$\Delta w_{Bn}^1 = \frac{\alpha v_{B0} t_{An}}{\sum_{n=1}^{N}(v_{Bn} + \alpha v_{B0} t_{An}) + (1-\alpha)v_{B0}} = \alpha v_{B0} t_{An}$$

$$S_{BA}^1 = \sum_{n=1}^{N} \Delta w_{Bn}^1 t_{An} = \alpha v_{B0} \sum_{n=1}^{N} t_{An} t_{An}$$

$$S_{BB}^1 = \sum_{n=1}^{N} \Delta w_{Bn}^1 t_{Bn} = \alpha v_{B0} \sum_{n=1}^{N} t_{An} t_{Bn} = 0$$

and Eq. (15) becomes

$$\sum_{n=1}^{N} (t_{An}, -, v_{An}) t_{An} < 0.$$

$\blacksquare$

Proof of Proposition 2. Assume that a percentage α, $\alpha \leq 1$, of voting stock v_{B0} is turned into preferred stock. As a result of the switch from voting stock to preferred stock, shareholder n owning a

percentage of voting stock v_{Bn} now owns a larger percentage, namely $v_{Bn}/(1-\alpha v_{B0})$. However, t_{Bn} and t_{B0} remain unchanged. Hence, after the divestiture:

$$v_{Bn}^2 = v_{Bn}/(1-\alpha v_{B0})$$
$$v_{B0}^2 = (1-\alpha)v_{B0}/(1-\alpha v_{B0}).$$

As a result of these changes there will be a variation in the weights given by firm B to each shareholder. This variation is given by

$$\Delta w_{Bn}^2 = \frac{f(v_{Bn}/(1-\alpha v_{B0}))}{\sum_{n=1}^{N} f(v_{Bn}/(1-\alpha v_{B0})) + f((1-\alpha)v_{B0}/(1-\alpha v_{B0}))}$$
$$- \frac{f(v_{Bn})}{\sum_{n=1}^{N} f(v_{Bn}) + f(v_{B0})}$$
$$= \frac{f(v_{Bn})}{\sum_{n=1}^{N} f(v_{Bn}) + f((1-\alpha)v_{B0})} - \frac{f(v_{Bn})}{\sum_{n=1}^{N} f(v_{Bn}) + f(v_{B0})} > 0.$$

As for firm A:

$$\Delta w_{B0}^2 = \frac{f((1-\alpha)v_{B0}/(1-\alpha v_{B0}))}{\sum_{n=1}^{N} f(v_{Bn}/(1-\alpha v_{B0})) + f((1-\alpha)v_{B0}/(1-\alpha v_{B0}))} - \frac{f(v_{B0})}{\sum_{n=1}^{N} f(v_{Bn}) + f(v_{B0})}$$
$$= \frac{f((1-\alpha)v_{B0})}{\sum_{n=1}^{N} f(v_{Bn}) + f((1-\alpha)v_{B0})} - \frac{f(v_{B0})}{\sum_{n=1}^{N} f(v_{Bn}) + f(v_{B0})} < 0,$$

where we have made use of the property: $f(v_{in})/f(v_{in'}) = f(\theta v_{in})/f(\theta v_{in'})$.

Thus, we have

$$k_{BA}^2 = \sum_{n=1}^{N} w_{Bn}^2 t_{An}^2 = \sum_{n=1}^{N} \left(w_{Bn} + \Delta w_{Bn}^2\right)t_{An} = k_{BA} + S_{BA}^2$$
$$k_{BB}^2 = \sum_{n=1}^{N} w_{Bn}^2 t_{Bn}^2 = \sum_{n=1}^{N} \left(w_{Bn} + \Delta w_{Bn}^2\right)t_{Bn} = k_{BB} + S_{BB}^2$$

with S_{BA}^2 and S_{BB}^2 non-negative. Hence,

$$K_{BA}^2 = k_{BA}^2 + w_{B0}^2 k_{AA}^2 = k_{BA} + S_{BA}^2 + w_{B0}^2 K_{AA} = K_{BA} + S_{BA}^2 + \Delta w_{B0}^2 K_{AA}$$
$$K_{BB}^2 = k_{BB}^2 + w_{B0}^2 k_{AB}^2 = K_{BB} + S_{BB}^2 + \Delta w_{B0}^2 K_{AB}$$

and Eqs. (11) and (13) become, respectively:

$$\gamma_A^2 - \gamma_A = t_{B0}^2 - t_{B0} + \frac{K_{AB}^2 - K_{AB}}{K_{AA}} = t_{B0} - t_{B0} + \frac{K_{AB} - K_{AB}}{K_{AA}} = 0$$
$$sign\left(\gamma_B^2 - \gamma_B\right) = sign\left(K_{BB}K_{BA}^2 - K_{BB}^2 K_{BA} + K_{BA}K_{BA}^2\left(t_{B0} - t_{B0}^2\right)\right)$$
$$= sign\left(K_{BB}K_{BA}^2 - K_{BB}^2 K_{BA}\right).$$

Therefore, Option 2 does not lead to any change in γ_A and it leads to a reduction in γ_B if and only if

$$K_{BB}K_{BA}^2 - K_{BB}^2 K_{BA} < 0 \Leftrightarrow k_{BB}k_{AA} - k_{BA}k_{AB} > 0.$$

∎

Proof of Proposition 3. After a complete divestiture of voting stock we will have $v_{B0}^3 = w_{B0}^3 = v_{B0}^4 = w_{B0}^4 = 0$ in both Options 3 and 4. There will also be a decrease in t_{B0} equal to $v_{B0}V_B$, i.e., $t_{B0}^3 = t_{B0}^4 = t_{B0} - v_{B0}V_B$.

In Option 3 we will have:

$$\Delta w_{Bn}^3 = \frac{f(v_{Bn})}{\sum_{n=1}^{N} f(v_{Bn}) + 0 + f(0)} - \frac{f(v_{Bn})}{\sum_{n=1}^{N} f(v_{Bn}) + f(v_{B0})} > 0$$
$$\Delta w_{B0}^3 = -w_{B0} = -\frac{f(v_{B0})}{\sum_{n=0}^{N} f(v_{Bn})} < 0$$

and

$$k_{BA}^3 = \sum_{n=1}^{N} w_{Bn}^3 t_{An}^3 = \sum_{n=1}^{N}\left(w_{Bn} + \Delta w_{Bn}^3\right)t_{An} = k_{BA} + S_{BA}^3$$
$$k_{BB}^3 = \sum_{n=1}^{N} w_{Bn}^3 t_{Bn}^3 = \sum_{n=1}^{N}\left(w_{Bn} + \Delta w_{Bn}^3\right)t_{Bn} = k_{BB} + S_{BB}^3$$

where S_{BA}^3 and S_{BB}^3 are non-negative. Hence,

$$K_{BA}^3 = k_{BA}^3 + w_{B0}^3 k_{AA}^3 = k_{BA} + S_{BA}^3 = K_{BA} + S_{BA}^3 + \Delta w_{B0}^3 k_{AA}$$
$$K_{BB}^3 = k_{BB}^3 + w_{B0}^3 k_{AB}^3 = k_{BB} + S_{BB}^3 = K_{BB} + S_{BB}^3 + \Delta w_{B0}^3 k_{AB}$$

and Eq. (11) becomes

$$\gamma_A^3 - \gamma_A = t_{B0}^3 - t_{B0} + \frac{K_{AB}^3 - K_{AB}}{K_{AA}} = (t_{B0} - v_{B0}V_B) - t_{B0} + \frac{K_{AB} - K_{AB}}{K_{AA}}$$
$$= -v_{B0}V_B$$

while Eq. (13) becomes

$$sign\left(\gamma_B^3 - \gamma_B\right) = sign\left(K_{BB}K_{BA}^3 - K_{BB}^3 K_{BA} + K_{BA}K_{BA}^3(t_{B0} - (t_{B0} - v_{B0}V_B))\right)$$
$$= sign\left(K_{BB}\left(S_{BA}^3 + \Delta w_{B0}^3 k_{AA}\right) - K_{BA}\left(S_{BB}^3 + \Delta w_{B0}^3 k_{AB}\right) + \right.$$
$$\left. + K_{BA}\left(K_{BA} + S_{BA}^3 + \Delta w_{B0}^3 k_{AA}\right)v_{B0}V_B\right).$$

Given that this is a complete divestiture we have $\Delta w_{B0}^3 = -w_{B0}$ and

$$sign\left(\gamma_B^3 - \gamma_B\right) = sign(K_{BB}\left(-w_{B0}k_{AA} + S_{BA}^3\right) - \left(-w_{B0}k_{AB} + S_{BB}^3\right)K_{BA} +$$
$$+ K_{BA}\left(K_{BA} - w_{B0}k_{AA} + S_{BA}^3\right)v_{B0}V_B)$$
$$= sign(w_{B0}(K_{BA}k_{AB} - K_{BB}k_{AA}) + K_{BB}S_{BA}^3 - K_{BA}S_{BB}^3 +$$
$$+ \left(k_{BA} + S_{BA}^3\right)K_{BA}v_{B0}V_B).$$

This is negative if and only if

$$-w_{B0}(k_{AA}k_{BB} - k_{AB}k_{BA}) + S_{BA}^3(k_{BB} + w_{B0}k_{AB}) - S_{BB}^3(k_{BA} + w_{B0}k_{AA}) + \qquad (16)$$
$$+ V_B v_{B0}\left(S_{BA}^3 + k_{BA}\right)(k_{BA} + w_{B0}k_{AA}) < 0$$

In the absence of common shareholders $k_{AB} = k_{BA} = S_{BA}^3 = 0$ and Eq. (16) becomes

$$-k_{AA}w_{B0}\left(k_{BB} + S_{BB}^3\right) < 0$$

which is always true.

In Option 4 there will be a new shareholder, shareholder $N+1$, which will take the role of firm A. Hence, $\Delta w_{Bn}^4 = 0$ for the N original shareholders and $\Delta w_{BN+1}^4 = w_{B0} > 0$. This will only affect k_{BB} which becomes:

$$k_{BB}^4 = \sum_{n=1}^{N+1} w_{Bn}^4 t_{Bn}^4 = \sum_{n=1}^{N}(w_{Bn} + 0)t_{Bn} + \left(0 + \Delta w_{BN+1}^4\right)t_{BN+1} = k_{BB} + w_{B0}v_{B0}V_B.$$

Thus,

$$K_{BA}^4 = k_{BA}^4 + w_{B0}^4 k_{AA}^4 = k_{BA}^4 = k_{BA}$$
$$K_{BB}^4 = k_{BB}^4 + w_{B0}^4 k_{AB}^4 = k_{BB}^4 = k_{BB} + w_{B0}v_{B0}V_B.$$

Hence, Eqs. (11) and (13) become

$$\gamma_A^4 - \gamma_A = t_{B0}^4 - t_{B0} + \frac{K_{AB}^4 - K_{AB}}{K_{AA}} = (t_{B0} - v_{B0}V_B) - t_{B0} + \frac{K_{AB} - K_{AB}}{K_{AA}} = -v_{B0}V_B$$
$$sign\left(\gamma_B^4 - \gamma_B\right) = sign\left(K_{BB}K_{BA}^4 - K_{BB}^4 K_{BA} + K_{BA}K_{BA}^4(t_{B0} - (t_{B0} - v_{B0}V_B))\right)$$
$$= sign(-w_{B0}(k_{AA}k_{BB} - k_{AB}k_{BA}) + (k_{BA} + w_{B0}k_{AA})(k_{BA} - w_{B0})v_{B0}V_B).$$

In the absence of common shareholders, $k_{AB} = k_{BA} = S^3_{BA} = 0$ and

$$sign\left(\gamma^4_B - \gamma_B\right) = -w_{B0}k_{AA}(k_{BB} + V_B v_{B0} w_{B0}) < 0.$$

■

Proof of Proposition 4. From Lemma 2, Option 2 is better than Option 1 if the two following conditions hold:

$$\gamma^2_A - \gamma^1_A = t^2_{B0} - t^1_{B0} + \frac{K^2_{AB} - K^1_{AB}}{K_{AA}} \leq 0 \tag{17}$$

$$sign\left(\gamma^2_B - \gamma^1_B\right) = sign\left(K^1_{BB}K^2_{BA} - K^2_{BB}K^1_{BA} + K^1_{BA}K^2_{BA}\left(t^1_{B0} - t^2_{B0}\right)\right) \leq 0. \tag{18}$$

Using the expressions for K_{ij} presented in the Proofs of Propositions 1 and 2 we have

$$\gamma^2_A - \gamma^1_A = t_{B0} - (1 - \alpha)t_{B0} + \frac{K_{AB} - (K_{AB} + \alpha t_{B0}K_{AA})}{K_{AA}} = 0$$

and

$$sign\left(\gamma^2_B - \gamma^1_B\right) = $$
$$sign\left(\left(K_{BB} + S^1_{BB} + \Delta w^1_{B0}K_{AB}\right)\left(K_{BA} + S^2_{BA} + \Delta w^2_{B0}K_{AA}\right) - \right.$$
$$\left. \left(K_{BB} + S^2_{BB} + \Delta w^2_{B0}K_{AB}\right)\left(K_{BA} + S^1_{BA} + \Delta w^1_{B0}K_{AA}\right)\right).$$

In the particular case of no-common shareholders we have $K_{BB} = k_{BB}$, $K_{BA} = w_{B0}k_{AA}$ and $K_{AB} = k_{AB} = S^2_{BA} = 0$:

$$sign\left(\gamma^2_B - \gamma^1_B\right) = sign\left(\left(\frac{k_{BB} + S^1_{BB}}{k_{BB} + S^2_{BB}}w^2_{B0} - w^1_{B0}\right)k_{AA} - S^1_{BA}\right).$$

We now show that the first term is positive, or that

$$\frac{k_{BB} + S^1_{BB}}{w^1_{B0}} > \frac{k_{BB} + S^2_{BB}}{w^2_{B0}} \iff \frac{w^1_{B0}}{\sum^N_{n=1} w^1_{Bn}t_{Bn}} < \frac{w^2_{B0}}{\sum^N_{n=1} w^2_{Bn}t_{Bn}}.$$

This is equivalent to

$$\frac{\overbrace{f((1-\alpha)v_{B0})}^{}}{\frac{\sum^N_{n=1} f(v_{Bn} + \alpha v_{B0}t_{An}) + f((1-\alpha)v_{B0})}{\sum^N_{n=1} f(v_{Bn} + \alpha v_{B0}t_{An})t_{Bn}}}{\frac{\sum^N_{n=1} f(v_{Bn} + \alpha v_{B0}t_{An}) + f((1-\alpha)v_{B0})}} < \frac{\overbrace{f((1-\alpha)v_{B0})}^{}}{\frac{\sum^N_{n=1} f(v_{Bn}) + f((1-\alpha)v_{B0})}{\sum^N_{n=1} f(v_{Bn})t_{Bn}}}{\frac{\sum^N_{n=1} f(v_{Bn}) + f((1-\alpha)v_{B0})}}.$$

After simplifying we obtain

$$\sum^N_{n=1} f(v_{Bn})t_{Bn} < \sum^N_{n=1} f(v_{Bn} + \alpha v_{B0}t_{An})t_{Bn}$$

which is always true.

Finally, if $f(v_{in}) = v_{in}$ and $\sum^N_{n=0} v_{Bn} = \sum^N_{n=1} t_{An} = 1$, we have

$$\Delta w^1_{B0} = -\alpha v_{B0}$$
$$\Delta w^1_{Bn} = \alpha t_{An} v_{B0}$$
$$S^1_{BB} = \sum^N_{n=1} \Delta w^1_{Bn}t_{Bn} = \alpha v_{B0} \sum^N_{n=1} t_{An}t_{Bn} = 0$$

and

$$\Delta w^2_{B0} = \frac{v_{B0} - \alpha v_{B0}}{1 - \alpha v_{B0}} - v_{B0}$$
$$\Delta w^2_{Bn} = \frac{v_{Bn}}{1 - \alpha v_{B0}} - v_{Bn}$$
$$S^2_{BB} = \sum^N_{n=1} \Delta w^2_{Bn}t_{Bn} = \frac{k_{BB}}{1 - \alpha v_{B0}} - k_{BB}.$$

Therefore,

$$sign\left(\gamma^2_B - \gamma^1_B\right) = sign\left(\left(\frac{k_{BB} + S^1_{BB}}{k_{BB} + S^2_{BB}}w^2_{B0} - w^1_{B0}\right)k_{AA} - S^1_{BA}\right) = sign\left(-S^1_{BA}\right) < 0.$$

■

Proof of Proposition 5. After a partial divestiture of voting stock we will have $v^3_{B0} = v^4_{B0} = (1 - \alpha)v_{B0}$ in both Options 3 and 4. There will also be a decrease in t_{B0} equal to $\alpha v_{B0}V_B$, i.e., $t^3_{B0} = t^4_{B0} = t_{B0} - \alpha v_{B0}V_B$.

In Option 3 we will have:

$$\Delta w^3_{Bn} = \frac{f(v_{Bn})}{\sum^N_{n=1} f(v_{Bn}) + 0 + f((1-\alpha)v_{B0})} - \frac{f(v_{Bn})}{\sum^N_{n=1} f(v_{Bn}) + f(v_{B0})} > 0$$
$$\Delta w^3_{B0} = \frac{f((1-\alpha)v_{B0})}{\sum^N_{n=1} f(v_{Bn}) + 0 + f((1-\alpha)v_{B0})} - \frac{f(v_{B0})}{\sum^N_{n=0} f(v_{Bn})} < 0$$

and k^3_{BA}, k^3_{BB}, K^3_{BA} and K^3_{BB} have the same expressions as in the proof of Proposition 3, with S^3_{BA} and S^3_{BB} non-negative.

In Option 4 there will be a new shareholder, shareholder $N + 1$, and:

$$\Delta w^4_{Bn} = \frac{f(v_{Bn})}{\sum^N_{n=1} f(v_{Bn}) + f(\alpha v_{B0}) + f((1-\alpha)v_{B0})} - \frac{f(v_{Bn})}{\sum^N_{n=0} f(v_{Bn})} > 0$$
$$\Delta w^4_{BN+1} = \frac{f(\alpha v_{B0})}{\sum^N_{n=1} f(v_{Bn}) + f(\alpha v_{B0}) + f((1-\alpha)v_{B0})} - 0 > 0$$
$$\Delta w^4_{B0} = \frac{f((1-\alpha)v_{B0})}{\sum^N_{n=1} f(v_{Bn}) + f(\alpha v_{B0}) + f((1-\alpha)v_{B0})} - \frac{f(v_{B0})}{\sum^N_{n=0} f(v_{Bn})} < 0$$

and

$$k^4_{BA} = \sum^{N+1}_{n=1} w^4_{Bn}t^4_{An} = \sum^N_{n=1}\left(w_{Bn} + \Delta w^4_{Bn}\right)t_{An} + (0 + \Delta w_{BN+1})0 = k_{BA} + S^4_{BA}$$
$$k^4_{BB} = \sum^{N+1}_{n=1} w^4_{Bn}t^4_{Bn} = \sum^N_{n=1}\left(w_{Bn} + \Delta w^4_{Bn}\right)t_{Bn} + (0 + \Delta w_{BN+1})t_{BN+1}$$
$$= k_{BB} + S^4_{BB} + w^4_{BN+1}\alpha v_{B0}V_B$$

with $S^4_{BA} > 0$.

Hence,

$$K^4_{BA} = k^4_{BA} + w^4_{B0}k^4_{AA} = K_{BA} + S^4_{BA} + \Delta w^4_{B0}k_{AA}$$
$$K^4_{BB} = k^4_{BB} + w^4_{B0}k^4_{AB} = K_{BB} + S^4_{BB} + \Delta w^4_{B0}k_{AB} + z$$

with $z = \alpha w^4_{BN+1}v_{B0}V_B$.

Using the expressions for the K's presented above, we have

$$\gamma^3_A - \gamma^4_A = t^3_{B0} - t^4_{B0} + \frac{K^3_{AB} - K^4_{AB}}{K_{AA}} = 0$$

and

$$sign\left(\gamma^3_B - \gamma^4_B\right) = sign\left(K^4_{BB}K^3_{BA} - K^3_{BB}K^4_{BA} + K^4_{BA}K^3_{BA}\left(t^4_{B0} - t^3_{B0}\right)\right)$$
$$= sign\left(K^4_{BB}K^3_{BA} - K^3_{BB}K^4_{BA}\right)$$
$$= sign\left(\left(K_{BB} + S^4_{BB} + \Delta w^4_{B0}k_{AB} + z\right)\left(K_{BA} + S^3_{BA} + \Delta w^3_{B0}k_{AA}\right)\right.$$
$$\left. - \left(K_{BB} + S^3_{BB} + \Delta w^3_{B0}k_{AB}\right)\left(K_{BA} + S^4_{BA} + \Delta w^4_{B0}k_{AA}\right)\right).$$

Without common shareholders we have $k_{AB} = k_{BA} = S^3_{BA} = S^4_{BA} = 0$ and, as long as it is not a complete sale:

$$sign\left(\gamma^3_B - \gamma^4_B\right) = sign\left(\frac{k_{BB} + S^4_{BB} + z}{k_{BB} + S^3_{BB}}\frac{w^3_{B0}}{w^4_{B0}} - 1\right) > 0.$$

We now show that

$$\frac{k_{BB} + S_{BB}^4 + z}{k_{BB} + S_{BB}^3} > \frac{w_{B0}^4}{w_{B0}^3}.$$

By definition, this is equivalent to

$$\frac{\sum_{n=1}^{N} w_{Bn}^4 t_{Bn}}{\sum_{n=1}^{N} w_{Bn}^3 t_{Bn}} + \frac{z}{k_{BB} + S_{BB}^3} > \frac{w_{B0}^4}{w_{B0}^3} \Longleftrightarrow$$

$$\frac{\frac{\sum_{n=1}^{N} f(v_{Bn}) t_{Bn}}{\sum_{n=1}^{N} f(v_{Bn}) + f(\alpha v_{B0}) + f((1-\alpha), v_{B0})}}{\frac{\sum_{n=1}^{N} f(v_{Bn}) t_{Bn}}{\sum_{n=1}^{N} f(v_{Bn}) + 0 + f((1-\alpha) v_{B0})}} + \frac{z}{k_{BB} + S_{BB}^3} > \frac{\frac{f((1-\alpha) v_{B0})}{\sum_{n=1}^{N} f(v_{Bn}) + f(\alpha v_{B0}) + f((1-\alpha), v_{B0})}}{\frac{f((1-\alpha) v_{B0})}{\sum_{n=1}^{N} f(v_{Bn}) + 0 + f((1-\alpha) v_{B0})}} \Longleftrightarrow$$

$$\frac{z}{k_{BB} + S_{BB}^3} > 0.$$

which is always true.

In case of a complete sale, i.e., $\Delta w_{B0}^4 = \Delta w_{B0}^3 = -w_{B0}$, even with common shareholders, one can show that $sign(\gamma_B^3 - \gamma_B^4) > 0$:

$$\left(K_{BB} + \Delta w_{B0}^4 k_{AB} + S_{BB}^4 + z\right)\left(K_{BA} + \Delta w_{B0}^3 k_{AA} + S_{BA}^3\right) -$$
$$- \left(K_{BB} + \Delta w_{B0}^3 k_{AB} + S_{BB}^3\right)\left(K_{BA} + \Delta w_{B0}^4 k_{AA} + S_{BA}^4\right)$$
$$= \left(k_{BB} + S_{BB}^4 + z\right)\left(k_{BA} + S_{BA}^3\right) - \left(k_{BB} + S_{BB}^3\right)\left(k_{BA} + S_{BA}^4\right) > 0.$$

In fact, $\frac{k_{BB} + S_{BB}^4}{k_{BB} + S_{BB}^3} + \frac{z}{k_{BB} + S_{BB}^3} > \frac{k_{BA} + S_{BA}^4}{k_{BA} + S_{BA}^3} \Longleftrightarrow$

$$\frac{\sum_{n=1}^{N} w_{Bn}^4 t_{Bn}}{\sum_{n=1}^{N} w_{Bn}^3 t_{Bn}} + \frac{z}{k_{BB} + S_{BB}^3} > \frac{\sum_{n=1}^{N} w_{Bn}^4 t_{An}}{\sum_{n=1}^{N} w_{Bn}^3 t_{An}} \Longleftrightarrow$$

$$\frac{\frac{\sum_{n=1}^{N} f(v_{Bn}) t_{Bn}}{\sum_{n=1}^{N} f(v_{Bn}) + f(\alpha v_{B0}) + f((1-\alpha) v_{B0})}}{\frac{\sum_{n=1}^{N} f(v_{Bn}) t_{Bn}}{\sum_{n=1}^{N} f(v_{Bn}) + 0 + f((1-\alpha) v_{B0})}} + \frac{z}{k_{BB} + S_{BB}^3} > \frac{\frac{\sum_{n=1}^{N} f(v_{Bn}) t_{An}}{\sum_{n=1}^{N} f(v_{Bn}) + f(\alpha v_{B0}) + f((1-\alpha) v_{B0})}}{\frac{\sum_{n=1}^{N} f(v_{Bn}) t_{An}}{\sum_{n=1}^{N} f(v_{Bn}) + 0 + f((1-\alpha) v_{B0})}}, \text{ which is always}$$

true.∎

Proof of Proposition 6. After a partial divestiture of voting stock we will have $v_{B0}^3 = (1-\alpha) v_{B0}$ and $t_{B0}^3 = t_{B0} - \alpha v_{B0} V_B$ under Option 3 and $v_{B0}^2 = (1-\alpha) v_{B0}/(1-\alpha v_{B0})$ and $t_{B0}^2 = t_{B0}$ under Option 2.

Before proceeding, note that

$$w_{Bn}^2 = \frac{f\left(\frac{v_{Bn}}{1-\alpha v_{B0}}\right)}{\sum_{n=1}^{N} f\left(\frac{v_{Bn}}{1-\alpha v_{B0}}\right) + f\left(\frac{(1-\alpha) v_{B0}}{1-\alpha v_{B0}}\right)} = \frac{f(v_{Bn})}{\sum_{n=1}^{N} f(v_{Bn}) + 0 + f((1-\alpha) v_{B0})} = w_{Bn}^3$$

$$w_{B0}^2 = \frac{f\left(\frac{(1-\alpha) v_{B0}}{1-\alpha v_{B0}}\right)}{\sum_{n=1}^{N} f\left(\frac{v_{Bn}}{1-\alpha v_{B0}}\right) + f\left(\frac{(1-\alpha) v_{B0}}{1-\alpha v_{B0}}\right)} = \frac{f((1-\alpha) v_{B0})}{\sum_{n=1}^{N} f(v_{Bn}) + 0 + f((1-\alpha) v_{B0})} = w_{B0}^3$$

which implies $\Delta w_{B0}^2 = \Delta w_{B0}^3$, $S_{BB}^2 = S_{BB}^3$ and $S_{BA}^2 = S_{BA}^3$.

From Eqs. (11) and (13) we have:

$$\gamma_A^2 - \gamma_A^3 = t_{B0}^2 - t_{B0}^3 + \frac{K_{AB}^2 - K_{AB}^3}{K_{AA}} = \alpha v_{B0} V_B > 0 \qquad (21)$$

and

$$sign\left(\gamma_B^2 - \gamma_B^3\right) = sign\left(K_{BB}^3 K_{BA}^2 - K_{BB}^2 K_{BA}^3 + K_{BA}^3 K_{BA}^2 \left(t_{B0}^3 - t_{B0}^2\right)\right)$$
$$= -\left(K_{BA} + \Delta w_{B0}^3 k_{AA} + S_{BA}^3\right)^2 (\alpha v_{B0} V_B) \leq 0.$$

In case of a complete divestiture, we have $(K_{BA} + \Delta w_{B0}^3 k_{AA} + S_{BA}^3)^2 = (K_{BA} - w_{B0} k_{AA} + S_{BA}^3)^2 = (k_{BA} + S_{BA}^3)^2$.

Assuming no common shareholders, this is 0 and $\gamma_B^2 - \gamma_B^3 = 0$: Option 3 is better for consumers because it leads to a lower γ_A. Otherwise, the weights each firm places on its rival always move in opposite directions.

Finally, after Option 1, we will have $v_{B0}^1 = (1-\alpha) v_{B0}$ and $t_{B0}^1 = (1-\alpha) t_{B0}$.

$$\gamma_A^1 - \gamma_A^3 = t_{B0}^1 - t_{B0}^3 + \frac{K_{AB}^1 - K_{AB}^3}{K_{AA}} = \alpha v_{B0} V_B \qquad (22)$$

and

$$sign\left(\gamma_B^1 - \gamma_B^3\right) = sign\left(K_{BB}^3 K_{BA}^1 - K_{BB}^1 K_{BA}^3 + K_{BA}^3 K_{BA}^1 \alpha(t_{B0} - V_B v_{B0})\right)$$
$$= sign\left(\left(K_{BB} + \Delta w_{B0}^3 k_{AB} + S_{BB}^3\right)\left(K_{BA} + S_{BA}^1 + \Delta w_{B0}^1 k_{AA}\right) + \right.$$
$$- \left(K_{BB} + S_{BB}^1 + \Delta w_{B0}^3 k_{AB}\right)\left(K_{BA} + \Delta w_{B0}^3 k_{AA} + S_{BA}^3\right) +$$
$$\left. - \left(K_{BA} + \Delta w_{B0}^3 k_{AA} + S_{BA}^3\right)\left(K_{BA} + S_{BA}^1 + \Delta w_{B0}^1 k_{AA}\right)\alpha(t_{B0} - V_B v_{B0})\right).$$

In the case of a complete divestiture $\Delta w_{B0}^3 = \Delta w_{B0}^1 = -w_{B0}$ and this expression becomes:

$$sign\left(\left(k_{BB} + S_{BB}^3\right)\left(k_{BA} + S_{BA}^1\right) - \left(k_{BB} + S_{BB}^1\right)\left(k_{BA} + S_{BA}^3\right)\right.$$
$$\left. \times \left(k_{BA} + S_{BA}^1\right)\alpha(t_{B0} - V_B v_{B0})\right).$$

If there are no common shareholders, we have $k_{BA} = S_{BA}^3 = 0$ and this simplifies to

$$\left(k_{BB} + S_{BB}^3\right) S_{BA}^1 > 0.$$

∎

Appendix B

Following Singh and Vives (1984) and assuming firm symmetry, let the representative consumer maximize

$$a(q_A + q_B) - \frac{1}{2}\left(bq_A^2 + bq_B^2 + 2dq_A q_B\right) - p_A q_A - p_B q_B$$

with $b > d$. This utility function leads to the following linear inverse demand functions:

$$p_A = a - bq_A - dq_B$$
$$p_B = a - bq_B - dq_A$$

or, equivalently, to the following direct demand functions:

$$q_A = \frac{a(b-d) - bp_A + dp_B}{b^2 - d^2}$$
$$q_B = \frac{a(b-d) - bp_B + dp_A}{b^2 - d^2}.$$

Assuming that marginal costs are equal to zero, it is straightforward to show that the equilibrium prices are given by:

$$p_A = \frac{a(b-d)(2b + d(\gamma_A + 1))}{4b^2 - d^2(\gamma_A + 1)(\gamma_B + 1)}$$
$$p_B = \frac{a(b-d)(2b + d(\gamma_A + 1))}{4b^2 - d^2(\gamma_A + 1)(\gamma_B + 1)}$$

where

$$\gamma_A = v_{B0}$$
$$\gamma_B = \frac{w_{B0} k_{AA}}{k_{BB} + k_{AA} w_{B0} v_{B0}}.$$

After a partial divestiture in which a fraction α of firm A's stock is sold to infinitesimal shareholders, we will have

$$v_{B0}^3 = (1-\alpha)v_{B0}$$
$$w_{B0}^3 = \frac{(1-\alpha)v_{B0}}{\sum_{n=1}^{N} v_{Bn} + (1-\alpha)v_{B0}} = \frac{(1-\alpha)v_{B0}}{1 - v_{B0} + (1-\alpha)v_{B0}} = \frac{(1-\alpha)v_{B0}}{1 - \alpha v_{B0}}$$
$$w_{Bn}^3 = \frac{v_{Bn}}{\sum_{n=1}^{N} v_{Bn} + (1-\alpha)v_{B0}} = \frac{v_{Bn}}{1 - \alpha v_{B0}}$$
$$k_{BB}^3 = \sum_{n=1}^{N} w_{Bn}^3 t_{Bn}^3 = \sum_{n=1}^{N} \frac{v_{Bn}}{1 - \alpha v_{B0}} t_{Bn} = \frac{k_{BB}}{1 - \alpha v_{B0}}.$$

and therefore,

$$\gamma_A^3 = (1-\alpha)v_{B0}$$
$$\gamma_B^3 = \frac{\frac{(1-\alpha)v_{B0}}{1-\alpha v_{B0}} k_{AA}}{\frac{k_{BB}}{1-\alpha v_{B0}} + k_{AA}\frac{(1-\alpha)v_{B0}}{1-\alpha v_{B0}}(1-\alpha)v_{B0}} = \frac{(1-\alpha)v_{B0}}{k + ((1-\alpha)v_{B0})^2}.$$

Writing $v = (1-\alpha)v_{B0}$ and inserting these expression into the equilibrium prices we obtain:

$$p_A = \frac{(k+v^2)(2b - d(v+1))(b-d)a}{4b^2(k+v^2) - d^2(v+1)(k+v+v^2)}$$
$$p_B = \frac{(2b(k+v^2) + d(k+v+v^2))(b-d)a}{4b^2(k+v^2) - d^2(v+1)(k+v+v^2)}$$

and

$$q_A = \frac{(2b^2(k+v^2) + db(1-v)(k+v^2) - d^2 v(k+v+v^2))}{4b^2(k+v^2) - d^2(v+1)(k+v+v^2)}\frac{a}{(b+d)}$$
$$q_B = \frac{(2b^2(k+v^2) + db(k-v+v^2) - d^2 v(v+1))}{4b^2(k+v^2) - d^2(v+1)(k+v+v^2)}\frac{a}{(b+d)}$$

or, letting $x = d/b < 1$ and normalizing prices and output

$$P_A = \frac{p_A}{a} = \frac{(k+v^2)(2 + x(v+1))(1-x)}{4(k+v^2) - x^2(v+1)(k+v+v^2)}$$
$$P_B = \frac{p_B}{a} = \frac{(2(k+v^2) + x(k+v+v^2))(1-x)}{4(k+v^2) - x^2(v+1)(k+v+v^2)}$$
$$Q_A = \frac{bq_A}{a} = \frac{(2(k+v^2) + x(1-v)(k+v^2) - x^2 v(k+v+v^2))}{4(k+v^2) - x^2(v+1)(k+v+v^2)}\frac{1}{(1+x)}$$
$$Q_B = \frac{bq_B}{a} = \frac{(2(k+v^2) + x(k-v+v^2) - x^2 v(v+1))}{4(k+v^2) - x^2(v+1)(k+v+v^2)}\frac{1}{(1+x)}.$$

In order to have non-negative prices and outputs we need that

$$g_1(k,v,x) = 4(k+v^2) - x^2(v+1)(k+v+v^2) > 0$$
$$g_2(k,v,x) = 2(k+v^2) + x(1-v)(k+v^2) - x^2 v(k+v+v^2) > 0$$
$$g_3(k,v,x) = 2(k+v^2) + x(k-v+v^2) - x^2 v(v+1) > 0.$$

Notice that $g_2(k,v,x) > 0$ is always true because:

$$\frac{\partial g_2(k,v,x)}{\partial k} = (x + 1 - vx + 1 - vx^2) > 0$$

and

$$g_2(0,v,x) = v^2(x+1)(1 - vx + 1 - x) > 0.$$

The other two conditions can be written respectively as

$$x < 2\sqrt{\frac{(k+v^2)}{(v+1)(k+v^2+v)}}$$
$$x < \frac{k - v + v^2 + \sqrt{(k+v^2-v)^2 + 8v(v+1)(k+v^2)}}{2v(v+1)}.$$

Normalized consumer surplus decreases with an infinitesimal divestiture if and only if $-\frac{\partial CS^*}{\partial \alpha}\big|_{\alpha=0} = \frac{\partial CS^*}{\partial v} > 0$. This has the same sign as

$$v^7 x(x+1)(2x - x^2 + 4) - 2(x+1)(2x - 2x^2 + x^3 + 4)v^6 +$$
$$6xk(x+2)(x+1)v^5 + (x+1)(2(2x - 2x^2 + x^3 + 4) + 3k(x-2)(x+2)^2)v^4 +$$
$$x(x+1)(2k(x+2)(3x-4) + (x^2 - 2x - 4) + k^2(6x + x^2 + 12))v^3 +$$
$$3k(x+2)(x^2 + x^3 + k(x^3 - 4x - 4))v^2 + 2xk(x+2)((k^2+1)(x+1)$$
$$+ k(x^2 - x - 4))v - k^2(x+2)^3(k+1).$$

Fig. 1, obtained numerically, presents the set of values in the (v,x)-space for which the expression above is positive, for arbitrary values of k, $k = 0$, $k = 0.075$ and $k = 0.100$. Fig. 1 also presents the only binding constraint needed to ensure positive outputs and prices, in gray, which corresponds to the case when $k = 0$.

References

Alley, W., 1997. Partial ownership arrangements and collusion in the automobile industry. J. Ind. Econ. 45, 191–205.

Bresnahan, Timothy, Salop, Steven, 1986. Quantifying the competitive effects of production joint ventures. Int. J. Ind. Organ. 4, 155–175.

Dietzenbacher, E., Smid, B., Volkerink, B., 2000. Horizontal integration in the Dutch financial sector. Int. J. Ind. Organ. 18, 1223–1242.

Dobson, Paul W., Waterson, Michael, 1997. Countervailing power and consumer prices. Econ. J. 107, 418–430.

Flath, David, 1992. Horizontal shareholding interlocks. Manag. Decis. Econ. 13, 75–77.

Foros, Ø., Kind, H.J., Shaffer, G., 2010. Mergers and partial ownership. The Bradley Policy Research Center, Financial Research and Policy Working Paper No. FR 10–11.

Gilo, D., Moshe, Y., Spiegel, Y., 2006. Partial cross ownership and tacit collusion. RAND J. Econ. 37, 81–99.

Hatton, C., Cardwell, D., 2000. Minority acquisitions. Compet. Law Insights 1, 5–6 (June).

Karle, H., Klein, T., Stahl, K.O., 2011. Ownership and control in a competitive industry. CEPR Discussion Paper No. 8277.

Malueg, D., 1992. Collusive behavior and partial ownership of rivals. Int. J. Ind. Organ. 10, 27–34.

Micola, A.R., Bunn, D.W., 2008. Crossholdings, concentration and information in capacity-constrained sealed bid-offer auctions. J. Econ. Behav. Organ. 66, 748–766.

O'Brien, Daniel P., Salop, Steven C., 2000. Competitive effects of partial ownership: financial interest and corporate control. Antitrust Law J. 67, 559–614.

Reitman, D., 1994. Partial ownership arrangements and the potential for collusion. J. Ind. Econ. 42, 313–322.

Reynolds, Robert J., Snapp, Bruce R., 1986. The competitive effects of partial equity interests and joint ventures. Int. J. Ind. Organ. 4, 141–153.

Singh, Nirvikar, Vives, Xavier, 1984. Price and quantity competition in a differentiated duopoly. RAND J. Econ. 15, 546–554.

Vives, Xavier, 2000. Oligopoly Pricing: Old Ideas and New Trends. MIT Press, Cambridge, MA.